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Discrete limit theorems for trigonometric polynomials

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1. Įvadas

Let, as usual, $\mathbb{R}$ and $\mathbb{C}$ denote the set of all real numbers and complex numbers, respectively, and let

$$p_n(t) = \sum_{j \leqslant n} a_j e^{i\lambda_j t}, \quad a_j \in \mathbb{C}, \quad \lambda_j \in \mathbb{R},$$

be a trigonometrical polynomial. It is known [1], that the polynomial $p_n(t)$ has a limit distribution low, i.e., the probability measure

$$\frac{1}{T} meas \{t \in [0, T], p_n(t) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}), \tag{1}$$

converges weakly to some measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$. Here $meas\{A\}$ is the Lebesgue measure of the set A, and $\mathcal{B}(S)$ stands for the class of Borel sets of the space S.

Instead of the measure (1) we may consider its discrete version. Let $h > 0$ be a fixed number. Define on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ the probability measure

$$P_N(A) = \frac{1}{N+1} \#(0 \leqslant m \leqslant N : p_n(mh) \in A)$$

and consider its weak convergence as $N \to \infty$. Here $\#A$ denotes the number of elements of the set A.

Theorem 1. *On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P such that the measure P_N converges weakly to P as $N \to \infty$.*

The latter theorem is a discrete version of a limit theorem for the measure (1).

Proof of Theorem 1. Clearly, instead of the measure P_N we can consider the measure

$$Q_N(A) = \frac{1}{N+1} \# \left(0 \leqslant m \leqslant N : (\operatorname{Re} p_n(mh), \operatorname{Im} p_n(mh)) \in A\right), \quad A \in \mathcal{B}(\mathbb{R}^2).$$

To prove the weak convergence of this measure it suffices to study the asymptotic behaviour of its characteristic function

$$\varphi_N(\tau_1, \tau_2) \stackrel{def}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\tau_1 x_1 + i\tau_2 x_2} dQ_N$$
$$= \frac{1}{N+1}\sum_{m=0}^{N} e^{i\tau_1 \operatorname{Re} p_n(mh) + i\tau_2 \operatorname{Im} p_n(mh)}. \tag{2}$$

It is well known that

$$\operatorname{Re} p_n(t) = \sum_{j=1}^{n} |a_j| \cos(\lambda_j t + \eta_j),$$

$$\operatorname{Im} p_n(t) = \sum_{j=1}^{n} |a_j| \sin(\lambda_j t + \eta_j).$$

Here $\eta_j = \arg a_j$. Let $J_m(x)$ stand for the Bessel functions. Since [1]

$$e^{ix\sin\theta} = \sum_{m=-\infty}^{\infty} J_m(x) e^{im\theta},$$

$$e^{ix\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(x) e^{im\theta},$$

we have that

$$e^{i\tau_1 \operatorname{Re} p_n(mh)} = \sum_{k_1,\dots,k_n=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)\dots J_{k_n}(|a_n{}'\tau_1)$$
$$\times \exp\left\{ i\left(\frac{\pi}{2}\sum_{j=1}^{n} k_j + mh\sum_{j=1}^{n} k_j\lambda_j + \sum_{j=1}^{n} k_j\eta_j \right)\right\}$$
$$= \sum_{k_1,\dots,k_n=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)\dots J_{k_n}(|a_n|\tau_1)$$
$$\times \exp\left\{ i\left(\sum_{j=1}^{n} k_j\left(\frac{\pi}{2} + \eta_j\right) + mh\sum_{j=1}^{n} k_j\lambda_j \right)\right\}, \tag{3}$$

$$e^{i\tau_2 \operatorname{Im} p_n(mh)} = \sum_{l_1,\dots,l_n=-\infty}^{\infty} J_{l_1}(|a_1|\tau_2)\dots J_{l_n}(|a_n|\tau_2)$$
$$\times \exp\left\{ i\left(\sum_{j=1}^{n} l_j\eta_j + mh\sum_{j=1}^{n} l_j\lambda_j \right)\right\}. \tag{4}$$

Let k be an integer, and let

$$\varphi(\tau_1,\tau_2) = \sum_{\substack{k_1,\dots,k_n=-\infty\\(k_1+l_1)\lambda_1+\dots+(k_n+l_n)\lambda_n=\frac{2k\pi}{h}}}^{\infty} \sum_{l_1,\dots,l_n=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)\dots J_{k_n}(|a_n|\tau_1) \times J_{l_1}(|a_1|\tau_2)\dots J_{l_n}(|a_n|\tau_2) \times \exp\Big\{i\Big(\frac{\pi}{2}\sum_{j=1}^{n}k_j + \sum_{j=1}^{n}(k_j+l_j)\eta_j\Big)\Big\}. \tag{5}$$

Then we find from (2)–(4) that

$$\varphi_N(\tau_1,\tau_2) = \varphi_n(\tau_1,\tau_2) + \sum_{\substack{k_1,\dots,k_n=-\infty\\(k_1+l_1)\lambda_1+\dots+(k_n+l_n)\lambda_n\neq\frac{2k\pi}{h}}}^{\infty} \sum_{l_1,\dots,l_n=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)\dots J_{k_n}(|a_n|\tau_1) \times J_{l_1}(|a_1|\tau_2)\dots J_{l_n}(|a_n|\tau_2)\exp\Big\{i\Big(\frac{\pi}{2}\sum_{j=1}^{n}k_j + \sum_{j=1}^{n}(k_j+l_j)\eta_j\Big)\Big\} \times\frac{1}{N+1}\sum_{m=0}^{N}\exp\Big\{imh\sum_{j=1}^{n}(k_j+l_j)\lambda_j\Big\}. \tag{6}$$

Obviously, if $(k_1+l_1)\lambda_1+\dots+(k_n+l_n)\lambda_n \neq \frac{2\pi k}{h}$,

$$\sum_{m=0}^{N}\exp\Big\{imh\sum_{j=1}^{n}(k_j+l_j)\lambda_j\Big\} = \frac{1-\exp\Big\{i(N+1)h\sum_{j=1}^{n}(k_j+l_j)\lambda_j\Big\}}{1-\exp\Big\{ih\sum_{j=1}^{n}(k_j+l_j)\lambda_j\Big\}}.$$

For the Bessel functions the estimate

$$J_k(z) = \frac{Bc_1^{|k|}}{|k|!}, \quad |z| \leqslant c_1,$$

is valid. Here $c_1, c_2, \dots$ are positive constants. Thus, taking an arbitrary $\varepsilon > 0$, we can find $K = K(\varepsilon)$ such that

$$\Bigg| \sum_{\substack{k_1,\dots,k_n=-\infty\\(|k_1|+|l_1|+\dots+|k_n|+|l_n|)>K(\varepsilon)}}^{\infty} \sum_{l_1,\dots,l_n=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)\dots J_{k_n}(|a_n|\tau_1) \times J_{l_1}(|a_1|\tau_2)\dots J_{l_n}(|a_n|\tau_2)$$

$$\times \exp\left\{i\left(\frac{\pi}{2}\sum_{j=1}^{n} k_j + \sum_{j=1}^{n}(k_j+l_j)\eta_j\right)\right\}$$

$$\times \frac{1}{N+1}\frac{1-\exp\left\{i(N+1)h\sum\limits_{j=1}^{n}(k_j+l_j)\lambda_j\right\}}{1-\exp\left\{ih\sum\limits_{j=1}^{n}(k_j+l_j)\lambda_j\right\}}\Bigg| < \frac{\varepsilon}{2} \tag{7}$$

for all $|\tau_1| \leqslant c_2$, $|\tau_2| \leqslant c_3$. Now are choose $N_0 = N_0(\varepsilon)$ such that for the remainder part of the sum in the equality (5) the inequality

$$\Bigg| \sum_{\substack{k_1,\ldots,k_n=-\infty \\ (|k_1|+|l_1|+\ldots+|k_n|+|l_n|)\leqslant K(\varepsilon)}}^{\infty} \sum_{l_1,\ldots,l_n=-\infty}^{\infty} J_{k_1}(|a_1|\tau_1)\ldots J_{k_n}(|a_n|\tau_1)$$

$$\times J_{l_1}(|a_1|\tau_2)\ldots J_{l_n}(|a_n|\tau_2)$$

$$\times \exp\left\{i\left(\frac{\pi}{2}\sum_{j=1}^{n} k_j + \sum_{j=1}^{n}(k_j+l_j)\eta_j\right)\right\}$$

$$\times \frac{1}{N+1}\frac{1-\exp\left\{i(N+1)h\sum\limits_{j=1}^{n}(k_j+l_j)\lambda_j\right\}}{1-\exp\left\{ih\sum\limits_{j=1}^{n}(k_j+l_j)\lambda_j\right\}}\Bigg| < \frac{\varepsilon}{2} \tag{8}$$

should be satisfied for $N \geqslant N_0$. Consequently, we obtained that $\varphi_N(\tau_1,\tau_2)$ converges to $\varphi(\tau_1,\tau_2)$ as $N \to \infty$ uniformly on τ_1 and τ_2 in each finite interval. Hence in view of the well known continuity theorem, see, for example [2] we obtain the weak convergence of the measure Q_N to the measure Q defined by the characteristic function $\varphi(\tau_1,\tau_2)$. The theorem is proved.

Now let G be a region on the complex plane, and let $H(G)$ denote the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Define a probability measure

$$Q_N(A) = \frac{1}{N+1}\#\left(0 \leqslant m \leqslant N : p_n(s+imh) \in A\right), \quad A \in \mathcal{B}(H(G)).$$

Theorem 2. *There exists a probability measure Q on $(H(G), \mathcal{B}(H(G)))$ such that the measure Q_N converges weakly to Q as $N \to \infty$.*

We begin the proof of Theorems 2 with the following lemma. Let γ be the unit circle on the complex plane, and let

$$\Omega_n = \prod_{l=1}^{n} \gamma_l,$$

where $\gamma_l = \gamma$ for all $l = 1, \dots, n$. Consider on $(\Omega_n, \mathcal{B}(\Omega_n))$ the probability measure

$$Q'_N(A) = \frac{1}{N+1} \# \left(0 \leqslant m \leqslant N : (\lambda_1^{imh}, \dots, \lambda_n^{imh}) \in A\right).$$

Lemma 1. *There exists a probability measure Q' on $(\Omega_n, \mathcal{B}(\Omega_n))$ such that the measure Q'_N converges weakly to Q' as $N \to \infty$.*

Proof. We have that the Fourier transforms $g_N(k_1, \dots, k_n)$ of the measure Q'_N (for the definition see [1]) is

$$g_N(k_1, \dots, k_n) = \frac{1}{N+1} \sum_{m=0}^{N} e^{imh \sum\limits_{l=1}^{n} k_l \lambda_l}.$$

Therefore, we have that

$$g_N(k_1, \dots, k_n) = \begin{cases} 1 & \text{if } h \sum\limits_{l=1}^{n} k_l \lambda_l = 2\pi r, r \in \mathbb{Z}, \\ \dfrac{1}{N+1} \dfrac{1 - e^{i(N+1)h \sum\limits_{l=1}^{n} k_l \lambda_l}}{1 - e^{ih \sum\limits_{l=1}^{n} k_l \lambda_l}} & \text{if } h \sum\limits_{l=1}^{n} k_l \lambda_l \neq 2\pi r. \end{cases}$$

Consequently,

$$\lim_{N \to \infty} g_n(k_1, \dots, k_n) = \begin{cases} 1 & \text{if } \sum\limits_{l=1}^{n} k_l \lambda_l = \frac{2\pi r}{h}, \\ 0 & \text{if } \sum\limits_{l=1}^{n} k_l \lambda_l \neq \frac{2\pi r}{h}. \end{cases}$$

Hence, by Theorem 1.3.19 from [1] the lemma follows.

Proof of Theorem 2. Define a function $h : \Omega_n \to H(G)$ by the formula

$$h(x_1, \dots, x_n) = \sum_{l=1}^{n} a_l e^{\lambda_l s} x_l^{\lambda_l mh}, (x_1, \dots, x_n) \in \Omega_n.$$

Clearly, the equality

$$\sum_{l=1}^{n} a_l e^{\lambda_l (s + imh)} = h(e^{i\lambda_1 mh}, \dots, e^{i\lambda_n mh})$$

holds, and the function h is continuous. Therefore from Lemma 2 and Theorem 5.1 from [2] we obtain that the measure $Q_N h^{-1}$ converges weakly to $Q' h^{-1}$ as $N \to \infty$.

Theorems 1 and 2 will be applied to prove discrete limit theorems for Dirichlet series.

References

[1] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-function*, Kluwer, Dordrecht, Boston, London (1996).
[2] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York (1968).

Diskrečioji ribinė teorema trigonometriniams polinomams

R. Kačinskaitė

Straipsnyje įrodomos dvi diskrečios ribinės teoremos trigometriniams polinomams tikimybinių matų silpno konvergavimo prasme.