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Asymptotic expansion for the distribution density function of the quadratic form of a stationary Gaussian process in the large deviation Cramer zone

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1. Formulation of result

Let $\{X_t,\ t=1,2,\dots\}$ be a real stationary Gaussian sequence with means $\mathbf{E}X_t=0$ and the covariance matrix (c.m.)

$$R=\Big[\mathbf{E}X_sX_t\Big]_{s=\overline{1,n}}^{t=\overline{1,n}},\qquad \det R\neq 0. \tag{1.1}$$

Denote

$$\xi_n=\sum_{s,t=1}^{n}a_{s,t}X_sX_t, \tag{1.2}$$

where, without loss of generality, we can suppose the matrix $A=\big[a_{s,t}\big]_{s=\overline{1,n}}^{t=\overline{1,n}}$ to be symmetric. We denote by $\mu_1,\mu_2,\dots,\mu_n$, a spectrum of eigenvalues of matrix RA obtained in the solution of the n^{th} degree algebraic equation $\det\big(A-\mu R^{-1}\big)=0$.

We know that the distribution of a r.v. ξ_n defined by equality (1.2) is the same as that of the r.v.

$$\eta_n=\sum_{j=1}^{n}\mu_jY_j^2, \tag{1.3}$$

where Y_j, $j=\overline{1,n}$ are independent Gaussian r.v's with $\mathbf{E}Y_j=0$ and $\mathbf{D}Y_j=\mathbf{E}Y_j^2=1$. Then

$$\mathbf{E}\xi_n=\mathbf{E}\eta_n=\sum_{j=1}^{n}\mu_j, \tag{1.4}$$

$$B_n^2=\mathbf{D}\xi_n=\mathbf{D}\eta_n=2\sum_{j=1}^{n}\mu_j^2. \tag{1.5}$$

Denote by

$$\widetilde{\xi}_n = \frac{\xi_n - \mathbf{E}\xi_n}{B_n}, \tag{1.6}$$

$$F_{\tilde{\xi}_n}(x) = \mathbf{P}(\widetilde{\xi}_n < x), \quad p_{\tilde{\xi}_n}(x) = \frac{d}{dx} F_{\tilde{\xi}_n}(x) \tag{1.7}$$

the distribution and the density function of the r.v. $\widetilde{\xi}_n$; and by

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)\,\mathrm{d}y, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} \exp\Big\{ -\frac{1}{2}x^2 \Big\} \tag{1.8}$$

the (0,1)-normal distribution and its density, respectively.

In order to obtain asymptotic expansions of the distribution function $F_{\widetilde{\xi}_n}(x)$ and its density $p_{\widetilde{\xi}_n}(x)$ of the r.v. $\widetilde{\xi}_n$, defined by equality (1.6), in large deviation zones, according to the general lemmas obtained by the author in [4], [1], one must have the estimates of the k^{th} order cumulants of the r.v. η_n

$$\Gamma_k(\eta_n) := \frac{1}{i^k}\frac{d^k}{dt^k} \ln f_{\eta_n}(t)\Big|_{t=0}, \qquad k = 1, 2, \dots, \tag{1.9}$$

where $f_\xi(t) = \mathbf{E}\exp\{it\xi\}$ is the characteristic function of the r.v. ξ.

Let $Z_j := \mu_j Y_j^2$, $j = 1, 2, \dots, n$. Recalling that Y_j - (0,1) are normal independent r. variables, we get

$$f_{Z_j}(t) = \mathbf{E}e^{itZ_j} = f_{Y_j^2}(\mu_j t) = \big(1 - 2i\mu_j t\big)^{-1/2},$$

$$f_{\eta_n}(t) = \prod_{j=1}^{n} \big(1 - 2i\mu_j t\big)^{-1/2}. \tag{1.10}$$

Then, by the definition of $\Gamma_k(\eta_n)$ and by equality (1.9), we obtain

$$\Gamma_k(\eta_n) = 2^{k-1}(k-1)! \sum_{j=1}^{n} \mu_j^k, \quad k = 1, 2, \dots. \tag{1.11}$$

Taking into account, that

$$\Gamma_1(\eta_n - \mathbf{E}\eta_n) = 0, \quad \Gamma_k(\eta_n - \mathbf{E}\eta_n) = \Gamma_k(\eta_n), \quad k = 2, 3, \dots,$$

we get

$$\Gamma_k(\widetilde{\xi}_n) = \Gamma_k\Big((\xi_n - \mathbf{E}\xi_n)/B_n\Big) = \Gamma_k(\eta_n)/B_n^k \tag{1.12}$$

$$= 2^{k-1}(k-1)! \sum_{j=1}^{n} \mu_j^k \Big/ \Big(2\sum_{j=1}^{n} \mu_j^2\Big)^{k/2}, \quad k = 2, 3, \dots. \tag{1.13}$$

Hence we obtain the following estimate of the k^{th} order cumulant $\Gamma_k\left(\widetilde{\xi}_n\right)$ of the r.v. $\widetilde{\xi}_n$:

$$|\Gamma(\widetilde{\xi}_n)| \leqslant (k-1)!/\Delta_n^{k-2}, \quad k = 2, 3, \ldots, \tag{1.14}$$

where

$$\Delta_n = \frac{B_n}{2 \max\limits_{1 \leqslant j \leqslant n} |\mu_j|} = \frac{\left(2 \sum\limits_{j=1}^{n} \mu_j^2\right)^{1/2}}{2 \max\limits_{1 \leqslant j \leqslant n} |\mu_j|}. \tag{1.15}$$

Next, let

$$\Delta_n^* := c_0 \Delta_n, \quad c_0 = (1/6)(\sqrt{2}/6), \tag{1.16}$$

$$T_n := \frac{1}{12}\Big(1 - \frac{x}{\Delta_n^*}\Big)\Delta_n^*, \tag{1.17}$$

θ_i, $i = 1, 2, \ldots$, stand for quantities not exceeding a unit in absolute value.

Theorem. *For the distribution density* $p_{\widetilde{\xi}_n}(x)$ *of the r.v.* $\widetilde{\xi}_n$ *defined by equality* (1.6) *in the interval*

$$0 \leqslant x < \Delta_n^*, \tag{1.18}$$

for integer l, $l \geqslant 1$, *the equality*

$$\frac{p_{\widetilde{\xi}_n}(x)}{\varphi(x)} = \exp\{L_n(x)\}\Bigg(1 + \sum_{\nu=0}^{l-1} M_{\nu,n}(x) + \theta_1 q(l)\Big(\frac{x+1}{\Delta_n^*}\Big)^l + \theta_2 \frac{2\pi e^2}{3} \frac{B_n}{\prod\limits_{k=1}^{4} |\mu_{i_k}|^{1/4}} \exp\Big\{-\frac{1}{5}T_n^2\Big\}\Bigg) \tag{1.19}$$

holds. Here

$$L_n(x) = \sum_{k=3}^{\infty} \lambda_{k,n} x^k \tag{1.20}$$

is a Cramer-Petrov series, where coefficients are found by formula (2.9) *in* [1] *expressed through the cumulants of the r.v.* $\widetilde{\xi}_n$*:*

$$\lambda_{3,n} = \frac{1}{3}\Gamma_3(\widetilde{\xi}_n),$$

$$\lambda_{4,n} = \frac{1}{24}\Big(\Gamma_4(\widetilde{\xi}_n) - 3\Gamma_3^2(\widetilde{\xi}_n)\Big),$$

$$\lambda_{5,n} = \frac{1}{120}\Big(\Gamma_5(\widetilde{\xi}_n) - 10\Gamma_3(\widetilde{\xi}_n)\Gamma_4(\widetilde{\xi}_n) + 15\Gamma_3^3(\widetilde{\xi}_n)\Big), \ldots$$

here the k^{th} order cumulant $\Gamma_k(\widetilde{\xi}_n)$, $k = 3, 4, \dots$, is expressed by formula (1.13). *Polynomials $M_{\gamma,n}(x)$ are expressed by formula* (6.8) [1], *where one must take a cumulant of the respective order of the r.v. $\widetilde{\xi}_n$ instead of r.v. ξ. In a special case,*

$$M_{0,n}(x) \equiv 0, \quad M_{1,n}(x) = -\frac{1}{2}\Gamma_3(\widetilde{\xi}_n)\, x,$$
$$M_{2,n}(x) = \frac{1}{8}\Big(5\Gamma_3^2(\widetilde{\xi}_n) - 2\Gamma_4(\widetilde{\xi}_n)\Big)x^2 + \frac{1}{24}\Big(3\Gamma_4(\widetilde{\xi}_n) - 5\Gamma_3^2(\widetilde{\xi}_n)\Big), \dots .$$

We get the expression of the quantity $q(l)$ from (6.11) [1], *supposing that $\gamma = 0$:*

$$q(l) = \left(\frac{3\sqrt{2e}}{2}\right)^l 8(l+2)^2 4^{3(l+1)} \Gamma\left(\frac{3l+1}{2}\right). \tag{1.21}$$

The quantities B_n, T_n and the function $\varphi(x)$ are defined by equalities (1.5), (1.17) *and* (1.8), *respectively.*

2. Proof of the theorem

Since, for the k^{th} order cumulant $\Gamma_k(\widetilde{\xi}_n)$, $k = 2, 3, \dots$, of the r.v. $\widetilde{\xi}_n$, estimate (1.14) holds, for the r.v. $\xi = \widetilde{\xi}_n$ the condition (S_γ) with $\gamma = 0$ and $\Delta = \Delta_n$, Δ_n being defined by equality (1.15), of Lemma 6.1 [1], [2] is satisfied. Based on this lemma we have to estimate the integral

$$R_n = \int\limits_{|t| \geqslant T_n} \left|f_{\widetilde{\eta}_n(h)}(t)\right| \mathrm{d}t, \tag{2.1}$$

where the quality T_n is defined by equality (1.17), and

$$\widetilde{\eta}_n(h) = (\eta_n(h) - M_n(h))/B_n(h), \tag{2.2}$$

$$\eta_n(h) = \sum_{j=1}^{n} Z_j(h), \tag{2.3}$$

In this turn $Z_j(h)$ is a r.v. $Z_j := \mu_j Y_j^2$, $j = 1, 2, \dots, n$, is a conjugate r.v. with the density function

$$p_{Z_j(h)}(x) = e^{hx} p_{Z_j}(x) \Big/ \int\limits_{-\infty}^{\infty} e^{hx} p_{Z_j}(x)\, \mathrm{d}x, \tag{2.4}$$

$$M_n(h) = \mathbf{E}\eta_n(h), \quad B_n^2(h) = \mathbf{D}\eta_n(h),$$

$$f_{\widetilde{\eta}_n(h)}(t) = \mathbf{E}\exp\left\{it\widetilde{\eta}_n(h)\right\} \tag{2.5}$$

is the characteristic function of the r.v. $\widetilde{\eta}_n(h)$.

Further, let

$$\varphi_{Z_j}(h) := \mathbf{E}e^{hZ_j} = \int_{-\infty}^{\infty} e^{hx} p_{Z_j}(x)\,\mathrm{d}x. \tag{2.6}$$

Since $f_{Z_j}(t) = \mathbf{E}e^{itZ_j} = \varphi_{Z_j}(it)$, taking into account the expression of $f_{Z_j}(t)$ by equality (1.10), we obtain

$$\varphi_{Z_j}(h) = \bigl(1 - 2\mu_j h\bigr)^{-1/2}, \quad j = 1, 2, \ldots, n. \tag{2.7}$$

Hence, basing on the expression of the density $p_{Z_{j(h)}}(x)$ of thr r.v. $Z_j(h)$ by equality (2.4), we get

$$f_{Z_{j(h)}}(t) = \frac{\varphi_{Z_j}(h+it)}{\varphi_{Z_j}(h)} = \Bigl(1 - 2\nu_j(h)it\Bigr)^{-1/2}, \tag{2.8}$$

where

$$\nu_j(h) = \mu_j / \bigl(1 - 2\mu_j h\bigr),\ j = 1, 2, \ldots, n. \tag{2.9}$$

Recalling that Y_j, $j = 1, 2, \ldots, n$, are independent (0,1) – Gaussian r.v's, we obtain

$$f_{\widetilde{\eta_n}(h)}(t) = \exp\left\{ -it\frac{M_n(h)}{B_n(h)} \right\} \prod_{j=1}^{n} f_{Z_j(h)}\bigl(t/B_n(h)\bigr). \tag{2.10}$$

From this, basing on the equality (2.8) we derive

$$\bigl|f_{\widetilde{\eta_n}(h)}(t)\bigr| = \prod_{j=1}^{n} \left(1 + \frac{4\nu_j^2(h)}{B_n^2(h)} t^2\right)^{-1/4}. \tag{2.11}$$

Recalling that the r.v. $\eta_n = \sum_{j=1}^{n} Z_j$, where $Z_j := \mu_j Y_j^2$, $j = 1, 2, \ldots, n$, are independent r.v's, we get

$$\varphi_{\eta_n}(h) = \mathbf{E}e^{h\eta_n} = \exp\left\{ \sum_{k=2}^{\infty} \frac{1}{k!} \Gamma_k(\eta_n) h^k \right\}. \tag{2.12}$$

Then the mean $M_n(h)$ and variance $B_n^2(h)$ of the r.v. $\eta_n(h)$ defined by equality (2.3) are equal to:

$$\begin{aligned} M_n(h) &= \frac{d}{dh} \ln \varphi_{\eta_n}(h) = \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \Gamma_k(\eta_n) h^{k-1}, \\ B_n^2(h) &= \frac{d^2}{dh^2} \ln \varphi_{\eta_n}(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \Gamma_k(\eta_n) h^{k-2}. \end{aligned} \tag{2.13}$$

respectively. Hence, basing on the expression of $\Gamma_k(\eta_n)$ by equality (1.12), we obtain

$$\begin{aligned} B_n^2(h) &= B_n^2\Big(1+\theta\sum_{k=3}^{\infty}(k-1)\Big(2\max_{1\leqslant j\leqslant n}|\mu_j|h\Big)^{k-2}\Big) \\ &= B_n^2(1+\theta(1/5)), \end{aligned} \tag{2.14}$$

for all $0 \leqslant h \leqslant \Delta_n/(12B_n)$, where B_n and Δ_n are defined by equalities (1.5) and (1.15), respectively. Now, recalling the definition of $\gamma_j(h)$ by equality (2.9) and the fact that $0 \leqslant h \leqslant (1/12)\Big(2\max_{1\leqslant j\leqslant n}|\mu_j|\Big)^{-1}$, we get

$$\nu_j(h) = \mu_j/(1-2\mu_j h) = \mu_j(1+(\theta/11)), \quad j=\overline{1,n}. \tag{2.15}$$

Next, using equalities(2.1) and (2.11), we have

$$\begin{aligned} R_n = &\int\limits_{|t|\geqslant T_n} \exp\Big\{-\frac{1}{4}\sum_{\substack{j=1\\ j\neq i_k}}^{n}\ln\Big(1+\frac{4\nu_j^2(h)}{B_n^2(h)}t^2\Big)\Big\} \\ &\times\prod_{k=1}^{4}\Big|f_{Z_{i_k}(h)}(t/B_n(h))\Big|\,dt. \end{aligned} \tag{2.16}$$

It is easy to check that

$$\prod_{k=1}^{2}\Big(1+\frac{4\nu_{i_k}^2(h)}{B_n^2(h)}t^2\Big) \geqslant \Big(1+\frac{4|\nu_{i_1}(h)\nu_{i_2}(h)|}{B_n^2(h)}t^2\Big)^2.$$

Consequently,

$$\prod_{k=1}^{2}\Big|f_{Z_{i_k}(h)}\Big(\frac{t}{B_n(h)}\Big)\Big| \leqslant \Big(1+\frac{4|\nu_{i_1}(h)\nu_{i_2}(h)|}{B_n^2(h)}t^2\Big)^{-1/2}. \tag{2.17}$$

Then

$$\int\limits_{-\infty}^{\infty}\prod_{k=1}^{2}\Big|f_{Z_{i_k}(h)}\Big(\frac{t}{B_n(h)}\Big)\Big|\,dt \leqslant \frac{\pi}{2}\Big(\frac{B_n^2(h)}{|\nu_{i_1}(h)\nu_{i_2}(h)|}\Big)^{1/2}. \tag{2.18}$$

Hence, making use of the Cauchy-Schwarz inequality, we obtain

$$\int\limits_{-\infty}^{\infty}\prod_{k=1}^{4}\Big|f_{Z_{i_k}(h)}\Big(\frac{t}{B_n(h)}\Big)\Big|\,dt \leqslant \frac{\pi}{2}\,\frac{B_n(h)}{\prod_{k=1}^{4}|\nu_{i_k}(h)|^{1/4}}. \tag{2.19}$$

Now, making use of equalities (2.14) and (2.15) one can easily check that $0 < 4\nu_j^2(h)T_n^2/B_n^2(h) < 1$. Thus, basing on the inequality $\ln(1+x) > \frac{1}{2}x$, $0 < x < 1$, we have

$$\ln\left(1 + \frac{4\nu_j^2(h)}{B_n^2(h)} T_n^2\right) \geqslant \frac{4}{5} \frac{2\mu_j^2}{B_n^2} T_n^2. \tag{2.20}$$

Hence, taking into account equalities (2.16) and (2.19), we obtain the estimate of integral R_n:

$$R_n \leqslant \frac{2\pi e^2}{3} \cdot \frac{B_n}{\prod\limits_{k=1}^{4} |\mu_{i_k}|^{1/4}} \exp\left\{ -\frac{1}{5} T_n^2 \right\}, \tag{2.21}$$

where T_n is defined by equality (1.17).

References

[1] Л. Саулис, В. Статулявичус, *Предельные Теоремы о Больших Уклонениях*, Мокслас, Вильнюс (1989).

[2] L. Saulis and V. Statulevičius, *Limit Theorems for Large Deviations*, Kluwer Academic Publishers, Dordrecht, Boston, London (1991).

[3] L. Saulis, Asymptotic expansions in large deviation zones for the distribution function of random variable with cumulants of regular growth, *Lithuanian Math. J.*, **36**, 365–392 (1996).

[4] Л. Саулис, Аппроксимация нормальным законом функции распределения и ее плотности нелинейного преобразования стационарного гаусовского процесса, *LMD mokslo darbai*, III tomas, MII, Vilnius, 489–498 (1999).

[5] J. Kubilius, *Probabilistic Methods in the Theory of Numbers*, American Mathematical Society, Providence (1964).

Stacionaraus Gauso proceso kvadratinės formos pasiskirstymo tankio funkcijos asimptotinis skleidinys didžiųjų nuokrypių Kramero zonoje

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Darbe gautas kvadratinės formos

$$\xi_n = \sum_{s,t=1}^{n} a_{s,t} X_s X_t, \quad \text{kur} \quad X_t,\ t = 1, 2, \ldots,$$

– stacionarus Gauso procesas ir $A = \left[a_{s,t}\right]_{s=\overline{1,n}}^{t=\overline{1,n}}$ – simetrinė matrica, pasiskirstymo tankio asimptotinis skleidinys didžiųjų nuokrypių Kramero zonoje. Šis rezultatas gautas, remiantis straipsnio autoriaus bendrąja lema 6.1 [1] ([2]), apjungianti kumulantų ir charakteristinių funkcijų metodus.