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Insulating sets and tensor products in Procesi category

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1. Introduction

All considered rings are associative with identity element which should be preserved by ring homomorphisms, all modules are unitary. Main results of this paper are that tensor product of centred monomorphisms is nonzero – Theorem 3.1, and that ideal maximal between disjoints with insulating set is strongly prime – Theorem 2.2. Insulating sets in rings first were introduced in [5], where the last result was announced.

2. Terminology. Main properties of insulating sets

By an ideal of a ring we understand a twosided ideal, and by (a) we denote ideal generated by the element (a).

Ring R is called *strongly prime* if it is prime and its central closure $Q(R)$ is a simple ring. See [1] and [2] for definitions and basic properties of the central closure and the extended centroid of a semiprime ring. An ideal of the ring is called strongly prime if corresponding factor ring is strongly prime.

The subring of $End_Z R$, acting from the left on R, generated as a ring by all left and right mutiplications l_a and r_b, where $a, b \in R$, is called a *multiplication ring* of the ring R and will be denoted by $M(R)$. So each $\lambda \in M(R)$ is of the form $\lambda = \sum_k l_{a_k} r_{b_k}$ where $a_k, b_k \in R$ and can be represented as the sum $\sum_k a_k \otimes b_k^\circ$, where $b_k^\circ \in R^\circ$ – ring opposite to R. Then $\lambda x = \sum_k a_k x b_k, x \in R$. Particularly, sending $\lambda \in M(R)$ to the $\lambda 1 = \sum_k a_k b_k$ gives canonical projection $\pi : M(R) \to R$ which is homomorphism of the left $M(R)$-modules.

Let M be an R-bimodule. Denote by $Z_M = Z_M(R) = \{\delta \in M \mid r\delta = \delta r, r \in R\}$ the set of R-centralizing elements of the M. A bimodule M is called *centred R-bimodule* if $M = RZ_M$.

Let $\varphi : R \to S$ be a ring homomorphism. We call φ the *centred homomorphism* if S is centred R-bimodule. It's easy to see that centred extension of the ring R is a factor ring of a polynomial ring over R. Rings and their centred homomophisms form a category, which is called *Procesi category*.

We call an element $a \in R$ a *symmetric zero divisor* if for each finite subset of elements $\{a_1, ..., a_n\} \subseteq (a)$, $Ann_{M(R)}\{a_1, ..., a_n\} \nsubseteq Ann_{M(R)}\{1_R\}$. Of course, when R is commutative, taking $n = 1, a_1 = a$, we obtain the usual definition of zero divisors.

For each prime ring R, we denote $F(R) = Z_{Q(R)}(Q(R))$ the centre of the central closure of R which is a field – the extended centroid of the R.

Let R be a ring. A finite set $A = \{a_1, ..., a_n\} \subseteq R$ is called an *insulator*, if

$$Ann_{M(R)}\{a_1, ..., a_n\} \subseteq Ann_{M(R)}\{1_R\};$$

i.e., if $\lambda a_1 = ... = \lambda a_n = 0$, implies $\lambda 1 = 0$. When R is commutative, set $\{a_1, ..., a_n\}$ is an insulator iff $Ann_R\{a_1, ..., a_n\} = 0$, or, in other words, ideal generated by elements $a_1, ..., a_n$ is dense.

In a semiprime ring R insulators can be characterised in terms of the central closure $Q(R)$ and extended centroid $F(R)$ of the ring. Indeed, using Theorem 32.3 in [9], we obtain the following

Proposition 2.1. *In a semiprime ring R finite set $A = \{a_1, ..., a_n\}$ is an insulator if and only if $1 \in AF(R)$, i.e., if*

$$a_1u_1 + ... + a_nu_n = 1$$

with suitable u_k, $(1 \leqslant k \leqslant n)$ from the extended centroid $F(R)$ of the ring R.

Let $\mathcal{I}$ be a set which elements are nonemptyfinite nonzero subsets of the ring R. We call the set $\mathcal{I}$ an *insulating set* if $\{1\} \in \mathcal{I}$, and for each $\{a_1, ..., a_n\} \in \mathcal{I}$ and elements $\lambda_1, ..., \lambda_m \in M(R)$ such that $\{\lambda_1 1, ..., \lambda_m 1\} \in \mathcal{I}$, we have that the set $\{\lambda_k a_l \mid 1 \leqslant k \leqslant m, 1 \leqslant l \leqslant n\} \in \mathcal{I}$.

EXAMPLE 1. The set $In(R)$, consisting of all insulators of the nonzero ring is insulating. Indeed, if $\lambda\lambda_k a_l = 0$ for all $1 \leqslant k \leqslant m$ and $1 \leqslant l \leqslant n$, we subsequently obtain that $\lambda\lambda_k 1 = \lambda b_k = 0$ for all $1 \leqslant k \leqslant m$, and $\lambda 1 = 0$.

EXAMPLE 2. Let $\mathfrak{p}$ be a strongly prime ideal of the ring R. Let $In(\mathfrak{p})$ be finite sets from R which are insulators modulo $\mathfrak{p}$. Its clear that $In(\mathfrak{p})$ is the insulating set.

We say that ideal $I \subset R$ is disjoint with an insulating set $\mathcal{I}$ if for each $A \in \mathcal{I}$, $A \not\subseteq I$. So $a \in R$ is a symmetric zero divisor iff (a) is disjoint with $In(R)$.

Theorem 2.2. *Let $\mathcal{I}$ be an insulating set in a ring R. Each ideal maximal between disjoints with $\mathcal{I}$ is strongly prime. Elements from $\mathcal{I}$ are insulators in $R/\mathfrak{p}$.*

Proof. Let $\mathfrak{p} \subset R$ be maximal ideal disjoint with $\mathcal{I}$. Such ideal exits because set of ideals disjoint with insulating set is nonempty and inductive. Let $x \notin \mathfrak{p}$. By maximality of $\mathfrak{p}$, ideal $\mathfrak{p} + (x)$ contains some element $A = \{a_1, ..., a_n\}$ from the $\mathcal{I}$. So $p_l + \mu_l x = a_l$ with suitable $p_l \in \mathfrak{p}$ and $\mu_l \in M(R)$. Analogously, if $\lambda 1 \notin \mathfrak{p}$, ideal $\mathfrak{p} + (\lambda 1)$ contains some $B = \{b_1, ..., b_m\} \in \mathcal{I}$, so $q_k + \nu_k \lambda 1 = b_k$, where $q_k \in \mathfrak{p}$, and $\nu_k \in M(R)$. Now $q_k + \nu_k \lambda 1 = (l_k + \nu_k \lambda)1$, where $l_k \in M(R)$ is the left multiplication by element q_k. So, by definition of the insulating set, some element $(l_k + \nu_k\lambda)(p_l + \mu_l x)$ is not in $\mathfrak{p}$. Thus $\lambda\mu_l x \notin \mathfrak{p}$. By Proposition 3.1 in [5], ideal $\mathfrak{p}$ is strongly prime.

By the definition, ideal $I \subset R$ is disjoint with $In(R)$ – the set of insulators of the ring R, iff I has the following property: for each finite subset of elements $\{i_1, ..., i_n\} \subseteq I$, there exists $\lambda \in M(R)$, such that $\lambda i_1 = ... = \lambda i_n = 0$ and $\lambda 1 \neq 0$. This remark and previous theorem immediately imply the following results.

Theorem 2.3. *Let $\varphi : R \longrightarrow S$ be a centred monomorphism, $I \subset R$ an ideal disjoint with the $In(R)$, then $I^e \cap R$ is also disjoint with $In(R)$, where $I^e = SIS = IZ_S(R)$ is extension of I in S. If $\mathfrak{p} \subset R$ is maximal between disjoint with $In(R)$, so it is strongly prime and $\mathfrak{p}^e \cap R = \mathfrak{p}$.*

Corollary 2.4. *Each symmetric zero divisor of the ring is contained in some strongly prime ideal which does not contain an insulator.*

3. Tensor products in Procesi category

It well known that Procesi category is closed under tensor products. Let $\varphi_\alpha : R \longrightarrow S, \alpha \in \mathfrak{I}$ be a family of centred monomorphisms. When R is commutative, it is standart fact that $\otimes_R S_\alpha, \alpha \in \mathfrak{I}$ is nonzero. For noncommutative R this still was unknown.

Let $\mathfrak{p} \subset R$ be maximal between the ideals disjoint with the set of insulators. We have shown that $\mathfrak{p}$ is strongly prime and that $\mathfrak{p}^e \cap R = \mathfrak{p}$ for the extensions of $\mathfrak{p}$ in each ring S_α. So we redused the question to the centred monomorphisms with the strongly prime ring R. By Theorem 2.7 in [5] central closure $Q(R)$ of the strongly prime ring is left and right flat as R-module. This crucial fact gives the reduction to the tensoring centred monomorphisms over the simple ring $Q(R)$. But it is known that centred extension over the simple ring is free module, generated by centralizing elements. So tensor product of the free nonzero modules is nonzero. Thus we have proved the following fact.

Theorem 3.1. *Tensor product of monomorphisms in Procesi category over nonzero ring is nonzero.*

Let's look at the main part of the given proof when R is commutative. We proved that ideal maximal between not containing finitely generated dense ideal is prime and is a contraction of its extension under monomorphisms. Namely these ideas could be generalized to the noncommutative case.

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Izoliuojančios sistemos ir tenzorinės sandaugos Pročezi kategorijoje

A. Kaučikas

Įrodyta, kad Pročezio kategorijoje tenzorinė monomorfizmų sandauga nelygi nuliui. Maksimalus idealas, nesikertantis su izoliuojančia sistema, yra stipriai pirminis.