

On statistical experiments observing H -diffusions

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1. Introduction

Let Θ be an arbitrary set, $h_\theta(x)$ and $g_\theta(x)$, $x \in (l, r) \subseteq \mathbb{R}^1$, $\theta \in \Theta$, be strictly positive measurable functions such that $h_\theta(x)g_\theta(x) =: \sigma^{-2}(x)$, $x \in (l, r)$, is independent of θ and satisfying the following assumptions:

$$\int_l^r h_\theta(x) dx = 1, \quad \theta \in \Theta, \quad (1)$$

for all $\theta \in \Theta$, $g_\theta(x)$ is differentiable in x , for all $x \in (l, r)$ and $\theta \in \Theta$ there exists $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \subset (l, r)$, such that

$$\int_{x-\varepsilon}^{x+\varepsilon} |g'_\theta(v)| dv < \infty, \quad (2)$$

and for some fixed $x_0 \in (l, r)$ and all $\theta \in \Theta$

$$G_\theta(x) := \int_{x_0}^x g_\theta(v) dv \rightarrow \infty \text{ as } x \uparrow r, \quad G_\theta(x) \rightarrow -\infty \text{ as } x \downarrow l. \quad (3)$$

Let $\Omega := C_{[0, \infty)}(l, r)$ be the space of all continuous mappings $\omega(\cdot) : [0, \infty) \rightarrow (l, r)$ with the topology of uniform convergence on compact sets, $\mathcal{F} := \mathcal{B}(\Omega)$ be a σ -algebra of the Borel subsets of Ω , $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ be the standard filtration of sub- σ -algebras of \mathcal{F} and $X(t, \omega) = \omega(t)$, $t \geq 0$, $\omega \in \Omega$.

Denoting $H_\theta(dx) = h_\theta(x)dx$ and

$$\mu_\theta(x) = -\frac{1}{2}(\ln g_\theta(x))'\sigma^2(x), \quad x \in (l, r), \theta \in \Theta, \quad (4)$$

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it is known (see, e.g., Mandl [7], Engelbert and Schmidt [1], Karatzas and Shreve [4], Grigelionis [2]) that under the assumptions (1)–(3), for each $\theta \in \Theta$, there exists unique probability measure P_θ on \mathcal{F} such that $P_\theta \circ X^{-1}(0) = H_\theta$,

$$M_\theta(t) := X(t) - \int_0^t \mu_\theta(X(s)) \, ds, \quad t \geq 0,$$

is a (P_θ, \mathbb{F}) -local martingale satisfying

$$\langle M_\theta \rangle_t = \int_0^t \sigma^2(X(s)) \, ds, \quad t \geq 0,$$

and X is a (P_θ, \mathbb{F}) - (H_θ) -diffusion, i.e., a strictly stationary diffusion process on an open interval $(l, r) \subseteq \mathbb{R}^1$ with the predetermined stationary distribution H .

For any fixed \mathbb{F} -stopping time T we shall consider statistical experiments $\{\Omega, \mathcal{F}_T, (P_{\theta,T}, \theta \in \Theta)\}$, where $P_{\theta,T}$ are restrictions of P_θ to \mathcal{F}_T . Using known results on differentiability of probability measures, corresponding to the diffusion processes (see, e.g. Liptser and Shiryaev [6]), in Section 2 we shall derive formulas for the log-likelihood ratios in terms of the stationary density functions and the diffusion coefficients. In Section 3 we shall describe curved exponential families of H -diffusions and related sufficient statistics (cf. Küchler and Sørensen [5]). The results are illustrated by some examples, including well-known diffusion models in natural sciences and mathematical finance (see, e.g., Grigelionis [2], [3]).

2. Log-likelihood ratios for H -diffusions

Proposition 1. a) *Let the assumptions (1)–(3) are fulfilled, $\theta_0 \in \Theta$ is fixed,*

$$P_\theta \left\{ \int_0^T \left[\left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \right]^2 ds < \infty \right\} = 1, \tag{5}$$

and

$$P_{\theta_0} \left\{ \int_0^T \left[\left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \right]^2 ds < \infty \right\} = 1, \quad \theta \in \Theta. \tag{6}$$

Then $P_{\theta,T} \sim P_{\theta_0,T}$ and

$$L_{\theta,T} := \ln \frac{dP_{\theta,T}}{dP_{\theta_0,T}} = \ln \frac{h_\theta(X(0))}{h_{\theta_0}(X(0))} + \frac{1}{2} \int_0^T \left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' dM_{\theta_0}(s)$$

$$-\frac{1}{8} \int_0^T \left[\left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \right]^2 \sigma^2(X(s)) \, ds. \quad (7)$$

b) If, additionally, the functions $g_\theta(x)$, $x \in (l, r)$, $\theta \in \Theta$, are twice continuously differentiable in x , then (cf. Morton [8])

$$\begin{aligned} L_{\theta, T} &= \frac{1}{2} \ln \frac{h_\theta(X(T))h_\theta(X(0))}{h_{\theta_0}(X(T))h_{\theta_0}(X(0))} \\ &\quad - \frac{1}{4} \int_0^T \left\{ \left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)'' + \frac{1}{2} \left[\left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \right]^2 \right. \\ &\quad \left. + \ln \left(\frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \left[\ln (h_{\theta_0}(X(s))\sigma^2(X(s))) \right]' \right\} \sigma^2(X(s)) \, ds. \end{aligned} \quad (8)$$

Proof. a) Applying the well-known criteria on differentiability of probability measures, corresponding to diffusion processes, and having in mind (4), it is enough to check, that

$$\frac{\mu_\theta(x) - \mu_{\theta_0}(x)}{\sigma^2(x)} = -\frac{1}{2} \left(\ln \frac{g_\theta(x)}{g_{\theta_0}(x)} \right)' = \frac{1}{2} \left(\ln \frac{h_\theta(x)}{h_{\theta_0}(x)} \right)', \quad x \in (l, r).$$

b) Because $\ln g_\theta(x) = -\ln(h_\theta(x)\sigma^2(x))$, from Ito's formula we find that

$$\begin{aligned} \frac{1}{2} \int_0^T \left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \, dM_{\theta_0}(s) &= \frac{1}{2} \ln \frac{h_\theta(X(T))h_{\theta_0}(X(0))}{h_{\theta_0}(X(T))h_\theta(X(0))} \\ &\quad - \frac{1}{4} \int_0^T \left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)' \left[\ln (h_{\theta_0}(X(s))\sigma^2(X(s))) \right]' \sigma^2(X(s)) \, ds \\ &\quad - \frac{1}{4} \int_0^T \left(\ln \frac{h_\theta(X(s))}{h_{\theta_0}(X(s))} \right)'' \sigma^2(X(s)) \, ds. \end{aligned} \quad (9)$$

Now from (7) and (9) we obtain (8).

Remark. If for all $\theta \in \Theta$ $g'_\theta(x)$, $x \in (l, r)$, is continuous in x and

$$P_\theta\{T < \infty\} = 1, \quad (10)$$

then, obviously, the assumptions (2), (5) and (6) are valid.

3. Curved exponential families of H -diffusions

Assume that there exist functions $\varphi_j(\theta)$, $\theta \in \Theta$, and $\psi_j(x)$, $x \in (l, r)$, $j = 1, \dots, d$, such that ψ_j , $j = 1, \dots, d$, are continuously differentiable, and

$$\ln \frac{h_\theta(x)}{h_{\theta_0}(x)} = \sum_{j=1}^d \varphi_j(\theta) \psi_j(x), \quad x \in (l, r), \quad \theta \in \Theta. \quad (11)$$

Proposition 2. a) Under the assumptions (1), (3), (10) and (11) the family $\{P_{\theta, T}, \theta \in \Theta\}$ is exponential and the statistics

$$\begin{aligned} \psi_j(X(0)) + \frac{1}{2} \int_0^T \psi'_j(X(s)) dM_{\theta_0}(s), \quad 1 \leq j \leq d, \\ \int_0^T \psi'_j(X(s)) \psi'_k(X(s)) \sigma^2(X(s)) ds, \quad j \leq k, 1 \leq j, k \leq d, \end{aligned}$$

are sufficient.

b) If, additionally, the functions $\psi_j(x)$, $x \in (l, r)$, $j = 1, \dots, d$, are twice continuously differentiable, then the family $\{P_{\theta, T}, \theta \in \Theta\}$ is exponential and the statistics

$$\begin{aligned} \psi_j(X(0)) + \psi_j(X(T)) - \frac{1}{2} \int_0^T \left\{ \psi''_j(X(s)) + \psi'_j(X(s)) \right. \\ \left. \times [\ln(h_{\theta_0}(X(s)) \sigma^2(X(s)))]' \right\} \sigma^2(X(s)) ds, \quad 1 \leq j \leq d, \\ \int_0^T \psi'_j(X(s)) \psi'_k(X(s)) \sigma^2(X(s)) ds, \quad j \leq k, 1 \leq j, k \leq d, \end{aligned}$$

are sufficient.

Proof. a) Indeed, from (7) and (11), we have that

$$\begin{aligned} L_{\theta, T} = \sum_{j=1}^d \varphi_j(\theta) \left(\psi_j(X(0)) + \frac{1}{2} \int_0^T \psi'_j(X(s)) dM_{\theta_0}(s) \right) \\ - \frac{1}{8} \sum_{j, k=1}^d \varphi_j(\theta) \varphi_k(\theta) \int_0^T \psi'_j(X(s)) \psi'_k(X(s)) \sigma^2(X(s)) ds, \end{aligned}$$

obviously proving a).

b) Analogously, from (8) and (11), we find that

$$\begin{aligned}
 L_{\theta, T} &= \sum_{j=1}^d \varphi_j(\theta) \frac{1}{2} \left(\psi_j(X(0)) + \psi_j(X(T)) \right. \\
 &\quad \left. - \frac{1}{2} \int_0^T \left\{ \psi_j''(X(s)) + \psi_j'(X(s)) [\ln(h_{\theta_0}(X(s))\sigma^2(X(s)))]' \right\} \sigma^2(X(s)) ds \right) \\
 &\quad - \frac{1}{8} \sum_{j,k=1}^d \varphi_j(\theta) \varphi_k(\theta) \int_0^T \psi_j'(X(s)) \psi_k'(X(s)) \sigma^2(X(s)) ds,
 \end{aligned}$$

proving b).

We conclude by considering several examples of parametric families of H -diffusions, for which the assumptions (1)–(3) are checked in Grigelionis [2], [3]. We shall denote norming constants for the stationary distributions by C and shall take $g_{\theta}(x) = (h_{\theta}(x)\sigma^2(x))^{-1}$, $x \in (l, r)$, $\theta \in \Theta$.

Example 1. Let $(l, r) = (0, 1)$,

$$\begin{aligned}
 h_{\theta}(x) &= Cx^{\beta_1-1}(1-x)^{\beta_2-1}e^{\lambda x}, \\
 \sigma^2(x) &= \sigma^2x^{\alpha_1}(1-x)^{\alpha_2}e^{\mu x}, \\
 \theta &= (\beta_1, \beta_2, \lambda) \in \Theta := (0, \infty) \cap [2 - \alpha_1, \infty) \times (0, \infty) \cap [2 - \alpha_2, \infty) \times R^1, \\
 \alpha_1, \alpha_2, \sigma^2 &> 0, \mu \in R^1 \text{ are known constants.}
 \end{aligned}$$

Because

$$\ln \frac{h_{\theta}(x)}{h_{\theta_0}(x)} = (\beta_1 - \beta_{10}) \ln x + (\beta_2 - \beta_{20}) \ln(1-x) + (\lambda - \lambda_0)x,$$

we find that $d = 3$, $\varphi_1(\theta) = \beta_1 - \beta_{10}$, $\varphi_2(\theta) = \beta_2 - \beta_{20}$, $\varphi_3(\theta) = \lambda - \lambda_0$, $\psi_1(x) = \ln x$, $\psi_2(x) = \ln(1-x)$, $\psi_3(x) = x$.

Example 2. a) Let $(l, r) = (0, \infty)$,

$$\begin{aligned}
 h_{\theta}(x) &= Cx^{\lambda-1} \exp \{ -\chi_1 x^{\beta_1} - \chi_2 x^{-\beta_2} \}, \\
 \sigma^2(x) &= \sigma^2 x^{\gamma}, \\
 \theta &= (\chi_1, \chi_2, \lambda) \in \Theta := (0, \infty) \times (0, \infty) \times R^1, \\
 \beta_1, \beta_2, \sigma^2 &> 0, \gamma \in R^1 \text{ are known constants.}
 \end{aligned}$$

Because

$$\ln \frac{h_{\theta}(x)}{h_{\theta_0}(x)} = (\lambda - \lambda_0) \ln x + (\chi_{10} - \chi_1)x^{\beta_1} + (\chi_{20} - \chi_2)x^{-\beta_2},$$

we have that $d = 3$, $\varphi_1(\theta) = \lambda - \lambda_0$, $\varphi_2(\theta) = \chi_{10} - \chi_1$, $\varphi_3(\theta) = \chi_{20} - \chi_2$, $\psi_1(x) = \ln x$, $\psi_2(x) = x^{\beta_1}$, $\psi_3(x) = x^{-\beta_2}$.

b) Let $(l, r) = (0, \infty)$,

$$h_\theta(x) = Cx^{\lambda-1} \exp \{-\chi_1 x^{\beta_1}\},$$

$$\sigma^2(x) = \sigma^2 x^\gamma,$$

$$\theta = (\chi_1, \lambda) \in \Theta := (0, \infty) \times [2 - \gamma, \infty) \cap (0, \infty),$$

$$\beta_1, \sigma^2 > 0, \gamma \in \mathbb{R}^1 \text{ are known constants.}$$

From the equality

$$\ln \frac{h_\theta(x)}{h_{\theta_0}(x)} = (\lambda - \lambda_0) \ln x + (\chi_{10} - \chi_1)x^{\beta_1}$$

we find that $d = 2$, $\varphi_1(\theta) = \lambda - \lambda_0$, $\varphi_2(\theta) = \chi_{10} - \chi_1$, $\psi_1(x) = \ln x$, $\psi_2(x) = x^{\beta_1}$.

c) Let $(l, r) = (0, \infty)$,

$$h_\theta(x) = Cx^{\lambda-1} \exp \{-\chi_2 x^{-\beta_2}\},$$

$$\sigma^2(x) = \sigma^2 x^\gamma,$$

$$\theta = (\chi_2, \lambda) \in \Theta := (0, \infty) \times (-\infty, 2 - \gamma] \cap (-\infty, 0),$$

$$\beta_2, \sigma^2 > 0, \gamma \in \mathbb{R}^1 \text{ are known constants.}$$

Because

$$\ln \frac{h_\theta(x)}{h_{\theta_0}(x)} = (\lambda - \lambda_0) \ln x + (\chi_{20} - \chi_2)x^{-\beta_2},$$

we obtain that $d = 2$, $\varphi_1(\theta) = \lambda - \lambda_0$, $\varphi_2(\theta) = \chi_{20} - \chi_2$, $\psi_1(x) = \ln x$, $\psi_2(x) = x^{-\beta_2}$.

Example 3. Let $(l, r) = (b, \infty)$,

$$h_\theta(x) = Cx^{-\theta},$$

$$\sigma^2(x) = \sigma^2(x - b)^\gamma,$$

$$\theta \in \Theta := (1, \infty) \cap [\gamma - 1, \infty),$$

$$b, \sigma^2 > 0, \gamma \geq 1 \text{ are known constants.}$$

In this case $d = 1$, $\varphi_1(\theta) = -\theta + \theta_0$ and $\psi_1(x) = \ln x$.

Example 4. Let $(l, r) = (0, \infty)$,

$$h_\theta(x) = Cx^\lambda \exp \{-\chi(\ln x)^{2k}\},$$

$$\sigma^2(x) = \sigma^2 x^\gamma,$$

$$\theta = (\chi, \lambda) \in \Theta := (0, \infty) \times \mathbb{R}^1,$$

$$\sigma^2 > 0, \gamma \in \mathbb{R}^1, k = 1, 2, \dots \text{ are known constants.}$$

We easily find that in this example $d = 2$, $\varphi_1(\theta) = \lambda - \lambda_0$, $\varphi_2(\theta) = -\chi + \chi_0$, $\psi_1(x) = \ln x$, $\psi_2(x) = (\ln x)^{2k}$.

Example 5. Let $(l, r) = (-\infty, \infty)$,

$$h_\theta(x) = C(1+x^2)^\lambda \exp\{-\kappa \arctg x\},$$

$$\sigma^2(x) = \sigma^2(1+x^2)^\gamma,$$

$$\theta = (\lambda, \kappa) \in \Theta := \left(-\infty, -\frac{1}{2}\right) \cap \left(-\infty, \frac{1}{2} - \gamma\right] \times \mathbb{R}^1,$$

$$\sigma^2 > 0, \gamma \in \mathbb{R}^1 \text{ are known constants.}$$

In this case $d = 2$, $\varphi_1(\theta) = \lambda - \lambda_0$, $\varphi_2(\theta) = -\kappa + \kappa_0$, $\psi_1(x) = \ln(1+x^2)$, $\psi_2(x) = \arctg x$.

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Apie statistinius eksperimentus stebint H -difuzijas

B. Grigelionis

Darbe yra nagrinėjama ergodiškų griežtai stacionarių difuzinių procesų atvirame intervale $(l, r) \subseteq \mathbb{R}^1$ su iš anksto duotu stacionariuoju skirstiniu H , vadinamų H -difuzijomis, parametrinė šeima. Yra rastos formulės tikėtimumo santykių logaritmams stacionariųjų skirstinių tankių bei difuzijos koeficientų terminais, o taip pat aprašytos kreivos H -difuzijų eksponentinės šeimos ir atitinkamos pakankamos statistikos. Rezultatai yra iliustruojami keletu pavyzdžių iš gerai žinomų difuzinių modelių gamtos moksluose bei matematinėje finansų teorijoje.