Liet. matem. rink., T. 41, spec. nr., 88–93

The mean square of the Lerch zeta-function with respect to the parameter α

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The Lerch zeta-function $L(\lambda, \alpha, s)$, $s = \sigma + it$, for $\sigma > 1$, is defined by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. Here λ and $\alpha > 0$ are fixed real numbers. If λ is an integer number, then the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$, and if λ is not an integer, then $L(\lambda, \alpha, s)$ is analytically continuable to an entire function.

Let

$$I(s, \lambda) = \int_0^1 \left|L(\lambda, \alpha, s) - \alpha^{-s}\right|^2 d\alpha.$$

Denote by $\Gamma(s)$ the Euler gamma-function, and define the function $\tilde{\zeta}(\lambda, s)$, for $\sigma > 1$, by

$$\tilde{\zeta}(\lambda, s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

and by analytic continuation elsewhere. Let t_0 be an arbitrary positive number, and let B denote a number bounded by a constant.

Theorem. *Let $\frac{1}{2} < \sigma < 1$ be fixed and $t \geqslant t_0$. Then for any real λ*

$$\begin{aligned} I(\sigma + it, \lambda) = {} & \frac{1}{2\sigma - 1} + 2\Gamma(2\sigma - 1)\mathrm{Re}\left(\tilde{\zeta}(\lambda, 2\sigma - 1)\frac{\Gamma(1 - \sigma + it)}{\Gamma(\sigma + it)}\right) \\ & - 2\mathrm{Re}\,\frac{1}{1 - \sigma + it}\left(e^{-2\pi i \lambda}\tilde{\zeta}(\lambda, \sigma + it) - 1\right) + Bt^{-1}. \end{aligned}$$

The quantity $I(s, \lambda)$ was considered by Koksma and Lekkerkerker (1952), Balasubramanian (1979), Rane (1983), Sitaramachandrarao (1987), Zhang Wenpeng (1990, 1991, 1993), Katsurada (1992, 1998), Katsurada and Matsumoto (1994).

Proof of Theorem. Let, for brevity,

$$\widetilde{L}(\lambda, \alpha, s) = L(\lambda, \alpha, s) - \alpha^{-s}.$$

Then

$$\widetilde{L}(\lambda, \alpha, s) = \mathrm{e}^{2\pi i\lambda} L(\lambda, \alpha + 1, s),$$

and

$$\int_0^1 \widetilde{L}(\lambda, \alpha, u)\widetilde{L}(-\lambda, \alpha, v)\mathrm{d}\alpha = \int_1^2 L(\lambda, \alpha, u)L(-\lambda, \alpha, v)\,\mathrm{d}\alpha. \tag{1}$$

Suppose $\operatorname{Re} u > 1$ and $\operatorname{Re} v > 1$. Then

$$L(\lambda, \alpha, u)L(-\lambda, \alpha, v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{e}^{2\pi i\lambda(m-n)}(m+\alpha)^{-u}(n+\alpha)^{-v}. \tag{2}$$

Let $\mathbb{N}_0$ be the set of all non-negative integers. Denote by L a contour which separates the poles of the function

$$G(u, v, s; \lambda, \alpha) \overset{def}{=} \frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)}\widetilde{\zeta}(\lambda, -s)\zeta(u+v+s, \alpha)$$

at $s = 1 - u - v$, $-1 + n, n \in \mathbb{N}_0$, from the poles at $s = -u - n$, $n \in \mathbb{N}$. Let

$$g(u, v; \lambda, \alpha) = \frac{1}{2\pi i}\int_L G(u, v, s; \lambda, \alpha)\,\mathrm{d}s.$$

Suppose that $-\operatorname{Re} u < c < -1$. Then from properties of hypergeometric functions it follows that

$$(m+n+\alpha)^{-u}(n+\alpha)^{-v} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)} m^s (n+\alpha)^{-u-v-s}\,\mathrm{d}s. \tag{3}$$

Let

$$f(u, v; \lambda, \alpha) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \mathrm{e}^{2\pi i\lambda m}(m+n+\alpha)^{-u}(n+\alpha)^{-v}.$$

Then (2) can be written in the form

$$L(\lambda, \alpha, u)L(-\lambda, \alpha, v) = \zeta(u+v, \alpha) + f(u, v; \lambda, \alpha) + f(v, u; -\lambda, \alpha). \tag{4}$$

Since $c < -1$ and $\mathrm{Re}\,(u+v)+c > 1$, from (3) we find that

$$f(u,v;\lambda,\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(u,v,s;\lambda,\alpha)\,\mathrm{d}s.$$

Now we replace the line of integration by the contour L. This and the residue theorem yield

$$f(u,v;\lambda,\alpha) = \frac{\Gamma(u+v-1)\Gamma(1-v)}{\Gamma(u)}\tilde{\zeta}(\lambda,u+v-1) + g(u,v;\lambda,\alpha).$$

Consequently, by (4),

$$\begin{aligned} L(\lambda,\alpha,u)L(-\lambda,\alpha,u) = {} & \zeta(u+v;\alpha) + \Gamma(u+v-1)\Big(\tilde{\zeta}(\lambda,u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)} \\ & + \tilde{\zeta}(-\lambda,u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\Big) \\ & + g(u,v;\lambda,\alpha) + g(v,u;-\lambda,\alpha). \end{aligned}$$

Since, for $s \neq 1$,

$$\int_1^2 \zeta(s,\alpha)\mathrm{d}\alpha = \frac{1}{s-1}, \tag{5}$$

hence, for $\mathrm{Re}\,u > 1$, $\mathrm{Re}\,v > 1$, we obtain

$$\begin{aligned} \int_1^2 L(\lambda,\alpha,u)L(-\lambda,\alpha,v)\,\mathrm{d}\alpha = {} & \frac{1}{u+v-1}\Gamma(u+v-1)\Big(\tilde{\zeta}(\lambda,u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)} \\ & + \tilde{\zeta}(-\lambda,u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\Big) \\ & + \int_1^2 g(u,v;\lambda,\alpha)\,\mathrm{d}\alpha + \int_1^2 g(v,u;-\lambda,\alpha)\,\mathrm{d}\alpha. \end{aligned} \tag{6}$$

The well-known estimates of gamma-function show that the integral in the definition of $g(u,v;\lambda,\alpha)$ converges uniformly in $\alpha \in [1,2]$. Therefore, interchanging the order of integration and using (5), we find

$$\int_1^2 g(u,v;\lambda,\alpha)\mathrm{d}\alpha = \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\Gamma(u+s)}{\Gamma(u)}\tilde{\zeta}(\lambda,-s)\frac{\mathrm{d}s}{u+v+s-1}. \tag{7}$$

Let $\max(-\operatorname{Re} z, -1) < c < 0$, and $0 < \operatorname{Re} z < \operatorname{Re} \kappa$. Then the properties of hypergeometric functions imply the formula

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\kappa)\Gamma(z+w)\Gamma(1+w)\Gamma(-w)}{\Gamma(z)\Gamma(\kappa+1+w)} \mathrm{e}^{\pi i w} \mathrm{d}w = \frac{1}{\kappa - z}.$$

Therefore, taking, $-\operatorname{Re} u < c_0 < \min(-1, 1 - \operatorname{Re}(u+v))$ and $\max(-\operatorname{Re} u - c_0, -1) < b < 0$, and using (7), we have

$$\int_1^2 g(u, v; \lambda, \alpha)\, \mathrm{d}\alpha = -\frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \frac{\Gamma(-s)}{\Gamma(u)} \tilde{\zeta}(\lambda, -s) \frac{1}{2\pi i} \times \int_{b-i\infty}^{b+i\infty} \frac{\Gamma(1-v)\Gamma(u+w+s)\Gamma(1+w)\Gamma(-w)}{\Gamma(2-v+w)} \mathrm{e}^{\pi i w} \mathrm{d}w \mathrm{d}s. \tag{8}$$

It is not difficult to see that, for $\operatorname{Re} z > 1$ and $-\operatorname{Re} z < \sigma < -1$,

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(-s)\Gamma(s+z)}{\Gamma(z)} \tilde{\zeta}(\lambda, -s)\, \mathrm{d}s = \mathrm{e}^{-2\pi i \lambda} \tilde{\zeta}(\lambda, z) - 1. \tag{9}$$

If we suppose that $\operatorname{Re}(u+v) < 1$, then we may interchange the order of integration in (8). Thus, by (9), the equality (8) can be rewritten in the form

$$\int_1^2 g(u, v; \lambda, \alpha)\, \mathrm{d}\alpha = -\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\Gamma(1-v)\Gamma(u+w)\Gamma(1+w)\Gamma(-w)}{\Gamma(u)\Gamma(2-v+w)} \mathrm{e}^{\pi i w} \times \left(\frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \frac{\Gamma(-s)\Gamma(u+w+s)}{\Gamma(u+w)} \tilde{\zeta}(\lambda, -s)\, \mathrm{d}s \right) \mathrm{d}w$$

$$= -\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\Gamma(1-v)\Gamma(u+w)\Gamma(1+w)\Gamma(-w)}{\Gamma(u)\Gamma(2-v+w)} \mathrm{e}^{\pi i w} \left(\mathrm{e}^{-2\pi i \lambda} \tilde{\zeta}(\lambda, u+w) - 1 \right) \mathrm{d}w.$$

Now we shift the line of integration to the line $\operatorname{Re} w = b_1$, $0 < b_1 = b + 1$, and in view of the residue theorem we obtain

$$\int_1^2 g(u, v; \lambda, \alpha)\, \mathrm{d}\alpha = -\frac{\Gamma(1-v)}{\Gamma(2-v)} \left(\mathrm{e}^{-2\pi i \lambda} \tilde{\zeta}(\lambda, u) - 1 \right)$$

$$-\frac{1}{2\pi i} \int_{b_1-i\infty}^{b_1+i\infty} \frac{\Gamma(1-v)\Gamma(u+w)\Gamma(1+w)\Gamma(-w)}{\Gamma(u)\Gamma(2-v+w)} \mathrm{e}^{\pi i w} \left(\mathrm{e}^{-2\pi i \lambda} \tilde{\zeta}(\lambda, u+w) - 1 \right) \mathrm{d}w. \tag{10}$$

Now we insert the series $(\mathrm{Re}\,(u+w)>1)$

$$\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(1+m)^{u+w}}$$

in place of $e^{-2\pi i\lambda}\widetilde{\zeta}(\lambda,u+w)-1$ and integrate in (10) the term-by-term $(\mathrm{Re}\,(u+v)<1)$. Taking $w=s+1$, we find that the integral in (10) is

$$\frac{1}{2\pi i}\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+1)^{u}}\int\limits_{b-i\infty}^{b+i\infty}\frac{\Gamma(1-v)\Gamma(u+s+1)\Gamma(2+s)\Gamma(-s-1)}{\Gamma(u)\Gamma(3-v+s)}\left(\frac{e^{\pi i}}{m+1}\right)^{s+1}ds. \quad (11)$$

Let $F(a,b;c;z)$ denote the hypergeometric function. Then, for $|\arg(-z)|<\pi$ and $\max(-\mathrm{Re}\,a,-\mathrm{Re}\,b)<\sigma<0$, the formula

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;z)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^{s}ds$$

is valid. From this, letting $-z\to\frac{e^{\pi i}}{m+1}$, $0<\arg(-z)<\pi$, we find

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}F(a,b;c;-e^{\pi i})=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}e^{\pi is}ds.$$

Consequently, the expression (11) is equal to

$$-\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+1)^{u}}\frac{\Gamma(1-v)}{\Gamma(u)}\frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty}\frac{\Gamma(u+s+1)\Gamma(1+s)\Gamma(-s)}{\Gamma(3-v+s)}\left(\frac{e^{\pi i}}{m+1}\right)^{s+1}ds$$

$$=\frac{u}{(2-v)(1-v)}\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+1)^{u+1}}F\left(u+1,1;3-v;\frac{1}{m+1}\right). \quad (12)$$

From the definition of $F(a,b;c;z)$ by power series it follows that $F(a,b;c;z)\to 1$ as $z\to 0$. Therefore $F\left(u+1,1;3-v;\frac{1}{m+1}\right)\to 1$ as $m\to\infty$. Hence we have that the series in (12) converges absolutely for $\mathrm{Re}\,u>0$, and therefore it defines an analytic function of (u,v) in the region $\mathrm{Re}\,u>0$ and any v. Thus, by (10)–(12), for $0<\mathrm{Re}\,u<1$, $0<\mathrm{Re}\,v<1$,

$$\int\limits_{1}^{2}g(u,v;\lambda,\alpha)\,d\alpha=\frac{1}{v-1}(e^{-2\pi i\lambda}\widetilde{\zeta}(\lambda,u)-1)$$

$$-\frac{u}{(2-v)(1-v)}\sum_{m=1}^{\infty}\frac{e^{2\pi i\lambda m}}{(m+1)^{u+1}}F\left(u+1,1;3-v;\frac{1}{m+1}\right). \quad (13)$$

Similarly it can be obtained that, for $0 < \mathrm{Re}\, u < 1$, $0 < \mathrm{Re}\, v < 1$,

$$\int_1^2 g(v, u; -\lambda, \alpha)\, \mathrm{d}\alpha = \frac{1}{u-1}\left(\mathrm{e}^{2\pi i\lambda}\widetilde{\zeta}(-\lambda, v) - 1\right)$$
$$-\frac{v}{(2-u)(1-u)}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi i\lambda m}}{(m+1)^{v+1}}F\left(v+1, 1; 3-u; \frac{1}{m+1}\right). \tag{14}$$

Now the formulas (1), (6) and (13), (14), for $0 < \mathrm{Re}\, u < 1$, $0 < \mathrm{Re}\, v < 1$, yield

$$\int_0^1 \widetilde{L}(\lambda, \alpha, u)\widetilde{L}(-\lambda, \alpha, v)\, \mathrm{d}\alpha = \frac{1}{u+v-1}$$
$$+\Gamma(u+v-1)\left(\widetilde{\zeta}(\lambda, u+v-1)\frac{\Gamma(1-v)}{\Gamma(u)} + \widetilde{\zeta}(-\lambda, u+v-1)\frac{\Gamma(1-u)}{\Gamma(v)}\right)$$
$$+\frac{1}{v-1}\left(\mathrm{e}^{-2\pi i\lambda}\widetilde{\zeta}(\lambda, u) - 1\right) + \frac{1}{u-1}\left(\mathrm{e}^{2\pi i\lambda}\widetilde{\zeta}(-\lambda, v) - 1\right)$$
$$-\frac{u}{(2-v)(1-v)}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi i\lambda m}}{(m+1)^{u+1}}F\left(u+1, 1; 3-v; \frac{1}{m+1}\right)$$
$$-\frac{v}{(2-u)(1-u)}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi i\lambda m}}{(m+1)^{v+1}}F\left(v+1, 1; 3-u; \frac{1}{m+1}\right). \tag{15}$$

To obtain the theorem we take in (15) $u = \sigma + it$, $v = \sigma - it$. Clearly, the terms in (15) containing the series are estimated as Bt^{-1}. Therefore, the right-hand side of the equality of the theorem is obtained by using an obvious identity $z + \bar{z} = 2\mathrm{Re}\, z$.

Lercho dzeta funkcijos modulio kvadrato vidurkis parametro α atžvilgiu

A. Laurinčikas

Straipsnyje gauta Lercho dzeta funkcijos antrojo momento parametro atžvilgiu asimptotinė formulė, kai $t \to \infty$.