Liet. matem. rink., T. 41, spec. nr., 84–87

On the left strongly prime modules and ideals

Algirdas KAUČIKAS (VPU)
e-mail:al.kauchikas@vpu.lt

1. Introduction

All considered rings are associative with identity element which should be preserved by ring homomorphisms, all modules are unitary.

We give the new results on the theory of the one-sided (left) modules and ideals. Particularly, the conceptually new proof of the A.L. Rosenberg's theorem about one-sided strongly prime radical of the ring is given.

2. Left strongly prime modules and ideals

A left non-zero module M over the ring R is called *strongly prime* if for any non-zero $x, y \in M$, there exits a finite set of elements $\{a_1, \ldots, a_n\} \subseteq R$, $n = n(x, y)$, such that $Ann_R\{a_1x, \ldots, a_nx\} \subseteq Ann_R\{y\}$. A submodule N of some module M is called *strongly prime* if the quotient module M/N is strongly prime R-module. Particularly, a left ideal $\mathfrak{p} \subset R$ is called *strongly prime* if the quotient module $R/\mathfrak{p}$ is strongly prime R-module. See [7] for thurther investigation of the strongly prime submodules of the modules over commutative ring.

When the ring R is commutative, we immediately obtain, that if R-module M is strongly prime, then the annihilators of non-zero elements of M coincide and this common annihilator is the prime ideal of the ring R.

Indeed, if $x.y \in M$ are nonzero and $rx = 0$ for some $r \in R$, then, using the definition of the strongly prime module and commutativity of the ring R, we also obtain $ry = 0$, so $Ann_R\{x\} \subseteq Ann_R\{y\}$. Symmetrically $Ann_R\{y\} \subseteq Ann_R\{x\}$, so annhilators coincide. If $u, v \notin Ann_R\{x\}$, $x \neq 0$, then $y = ux \neq 0$, so $vy = vux \neq 0$, this gives $vu \notin Ann_R\{x\}$ and $Ann_R\{x\}$ is the prime ideal of the ring R.

Observe, that in the commutative case the clear inverse statement is true.

Theorem 2.1. *Let M be a non-zero module over a commutative ring R and $Ann_R\{x\} = Ann_R\{y\} = \mathfrak{p}$ for all non-zero elements $x, y \in M$. Then the annihilator $\mathfrak{p}$ is the prime ideal and M is strongly prime R-module.*

We omit the clear proof and return to the general situation.

Simple modules are evidently strongly prime, so maximal left ideals of the ring are the strongly prime ideals. Of course, annihilators of non-zero elements of the strongly prime module do not coinside in general but they are related in the following sense. Let L and K be a left ideals of the ring R. We say that $L \leqslant K$ if there exits the finite subset $A \subseteq R$ such that $(L : A) \subseteq K$, where

$$(L : A) = \{r \in R \,|\, rA \subseteq L\}$$

is the left ideal in R.

We call L and K equivalent if $L \leqslant K$ and $K \leqslant L$. Relation $\leqslant$ is an ordinary inclusion when R is commutative. Now it's easily seen from the definition, that annihilators of the non-zero elements of the strongly prime module are equivalent in the defined sense. More over, we obtain the generalization of the Theorem 2.1.

Theorem 2.2. *Let M be a non-zero left R-module M is strongly prime iff annihilators of its non-zero elements are equivalent.*

Proof. Indeed, if $A = \{a_1, \ldots, a_n\}$ and $(L : A) \subseteq K$, where $L = Ann_R\{x\}$, $K = Ann_R\{y\}$ for non-zero $x \cdot y \in M$, we obtain that $Ann_R\{a_1x, \ldots, a_nx\} = (L : A) \subseteq K = Ann_R\{y\}$, so M is strongly prime.

Now we characterize left strongly prime ideal $\mathfrak{p}$ of nonzero ring. By the definition applied to the element $u \notin R$ and $1 = 1_R \in R$, there exits the finite set $\{\alpha_1, \ldots, \alpha_n\} \subseteq R$, $n = n(u)$, having the following property: for each $v \notin \mathfrak{p}$ there exists $v\alpha_k u \notin \mathfrak{p}$ for some $1 \leqslant k \leqslant n$. Equivalently: if all $r\alpha_1 u, \ldots, r\alpha_n u \in \mathfrak{p}$, for some $r \in R$, then $r \in \mathfrak{p}$. Evidently, this property fully characterizes left strongly prime ideals of the ring in terms of the elements of the ring.

Let's look at the quasi-injective hull $Q(M)$ of the left strongly prime module M. By Theorem 19.2 in [2], $Q(M) = \Lambda\hat{M}$, where $\hat{M}$ is the injective hull of M and $\Lambda = End_R\hat{M}$.

The fact that $Q(M)$ is strongly prime R-module when M is strongly prime is known, see [1]. Denote by $\mathfrak{l}(R)$ the immage of the canonical ring homomorphism of R in $End_Z Q(M)$, obtained by sending $r \in R$ to the left multiplication $\mathfrak{l}_r \in End_Z Q(M)$, i.e., $\mathfrak{l}_r x = rx$ for $x \in Q(M)$. Let $H = End_R Q(M)$, elements of which we also write from the left. We also consider H as the canonical subring of $End_Z Q(R)$. Let $S = \mathfrak{l}(R)H$ be the subring of the ring $End_Z Q(M)$, acting on $Q(M)$ from the left. Now we put the definition of the strongly prime module in the most natural context.

Theorem 2.3. *A left R-module M is strongly prime iff its quasi-injective hull $Q(M)$ is the simple S-module.*

Proof. As mentioned above, when M is strongly prime R-module, $Q(M)$ also is strongly prime R-module. Let $x, y \in Q(M)$ are non-zero elements. Denote $z = (a_1x, \ldots, a_nx) \in Q(M)^n$, where elements $a_1x, \ldots, a_nx \in Q(M)$ are from the definition of the strongly prime R-module $Q(M)$. This exacly means, that the map $\varphi : Rz \to Ry$, sending rz to ry, $r \in R$ is the homomorphism of the R-modules. This homomorphism extends to the R-module homomorphism $\mathfrak{f} : \hat{M}^n \longrightarrow \hat{M}$. Let components of the $\mathfrak{f}$ are R-module homomorphisms $\mathfrak{f}_k : \hat{M} \longrightarrow \hat{M}$, $1 \leqslant k \leqslant n$. So we have $\mathfrak{f}z = \sum_k \mathfrak{f}_k a_k x = y$. But all $\mathfrak{f}_k a_k x \in Q(M)$, so, denoting by $\varphi_k \in End_R Q(M) = H$ of the homomorphisms $\mathfrak{f}_k$, $1 \leqslant k \leqslant n$, we obtain $y = \sum_k a_k \varphi_k x$. But $a_k \varphi_k \in S$, so $Q(M)$ is the simple S-module.

Theorem 2.4. *Let M_α, $\alpha \in \mathfrak{I}$ be a family of the left strongly prime R-modules, such that $Ann_R x_\alpha$ is equivalent to the $Ann_R x_\beta$ for all non-zero elements $x_\alpha \in M_\alpha$ and $x_\beta \in M_\beta$, $\alpha, \beta \in \mathfrak{I}$. Then the direct sum $\bigoplus_\alpha M_\alpha$, $\alpha \in \mathfrak{I}$ is the strongly prime R-module.*

Proof. We immediately reduce the proof to the finite direct sum, and, by induction, we can restrict to the case of two strongly prime R-modules M_1 and M_2. Let (x_1, x_2) and (y_1, y_2) be non-zero elements from the module $M_1 \bigoplus M_2$. Let, say, $x_1 \neq 0$, $\mathfrak{p} = Ann_R x_1$, and $A_1, A_2 \subseteq R$ be the finite sets of elements such that $rA_k x_k = 0$ implies $ry_k = 0$, $k \in \{1, 2\}$. We take $A_k = 1_R$ if one of $y_k = 0$. Taking $A = A_1 \bigcup A_2$, we have that if $r(x_1, x_2) = 0$, then, evidently, $rx_1 = 0$ and $r(y_1, y_2) = 0$, so $M_1 \bigoplus M_2$ is strongly prime.

From this we obtain the following exeptional fact for noncommutative rings.

Corollary 2.5. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be equivalent left strongly prime ideals of the ring R. Then their intersection $\mathfrak{q}$ is the left strongly prime ideal eguivalent to each $\mathfrak{p}_k$.*

Indeed modules $M_k = R/\mathfrak{p}_k$ are strongly prime, thus, by the Theorem above, module $M = \bigoplus_k M_k$ is strongly prime R-module. But $\mathfrak{q}$ is the annihilator of the element $(\bar{1}_R, \ldots, \bar{1}_R) \in M$, so $\mathfrak{q}$ is the left strongly prime ideal.

Corollary 2.6. *If R is classically semisimple ring, having only one class of isomorphic simple R-modules, then all proper left ideals of the ring R are strongly prime. Particularly, all proper left ideals of the matrix rings $M_n(D)$ over skew-field D are strongly prime.*

3. Left strongly prime radical of the ring

Intersection of all strongly prime ideals of the ring R *left strongly prime radical* of the ring R, which we denote by $rad_l R$. By A.L. Rosenberg's theorem (see [9]), $rad_l R$ coincides with Lewitzki radical $L(R)$ of the ring R. Unfortunately, A.L. Rosenberg's proof is very long and highly complicated. We give here the conceptual proof of this result.

Recall, that Lewitzki radical is the biggest locally nilpotent ideal of the ring. If some element $a \notin \mathfrak{p}$ for some left strongly prime ideal, there exist the finite set $s = \{a_1, \ldots . a_n\} \subseteq R$, such that $ra_1, \ldots, ra_n \in \mathfrak{p}$ implies $r \in \mathfrak{p}$. Evidently, $s^m \nsubseteq \mathfrak{p}$ for $m \in \mathbf{N}$, so $a \notin L(R)$.

Let now $a \notin L(R)$. This means, that there exist finite subset $s = \{a_1, \ldots, a_n\} \subseteq RaR$, which is not nilpotent. It's clear, that we may take the elements a_k in the form

$\alpha_k a \beta_k$ with $\alpha_k, \beta_k \in R$. Then the finite set $\bar{s} \subseteq Ra$, consisting of all elements of the form $\alpha_k a$ and $\beta_i \alpha_j a$ also is not nilpotent. Let $\bar{s} = r_1 a, \ldots, r_m a$. It's easy to check, that polynomial $F = (X_1 r_1 + \cdots + X_m r_m)a - 1$ is not left invertible in the polynomial ring $R\langle X_1, \ldots, X_m \rangle$. Thus the left ideal of the ring $R\langle X_1, \ldots, X_m \rangle$, generated by the polynomial F, is contained in some maximal ideal $\mathcal{M}$. Evidently $a \notin \mathcal{M}$. By the standart fact, $\mathcal{M} \cap R$ is the left strongly prime ideal of the ring R, which does not contain the given element a. Thus $rad_l R = L(R)$.

References

[1] J.A. Beachy, Some aspects of noncommutative localization, *Noncommutative Ring Theory*, LNM, **545**, Spriger-Verlag, Berlin, 2–31 (1975).

[2] C. Faith, *Algebra II, Ring Theory*, Springer–Verlag, Berlin (1976).

[3] D. Handelman, L. Lawrence, Strongly prime rings, *Trans. Amer. Math. Soc.*, **211**, 209–223 (1975).

[4] M. Hongan, On strongly prime modules and related topics, *Math. J.Okayama Univ.*, **24**, 117–132 (1982).

[5] P. Jara, P. Verhaege, A. Verschoren, On the left spectrum of a ring, *Comm. Algebra*, **22**(8), 2983–3002 (1994).

[6] A.Kaučikas, R.Wisbauer, On strongly prime rings and ideals, *Comm. Algebra*, **28**(11), 5461–5473 (2000).

[7] R.L. McCasland, M.E. Moore, P.F. Smith, On the spectrum of the module over a commutative ring, *Comm. Algebra*, **25**(1), 79–103 (1997).

[8] F. Van Oystaeyen, A. Verschoren, *Noncommutative Algebraic Geometry*, LNM, **887**, Springer-Verlag, Berlin (1981).

[9] A. Rosenberg, *Noncommutative Algebraic Geometry and Representations of Quantized Algebras*, Kluwer, Dordrecht (1995).

[10] R. Wisbauer, On prime modules and rings, *Comm. Algebra*, **11**, 2249–2265 (1983).

Apie stipriai pirminius kairiuosius modulius ir idealus

A. Kaučikas

Įrodyta, kad stipriai pirminio modulio kvaziinjektyvus apvalkalas yra paprastas modulis virš natūralaus jo endomorfizmų požiedžio. Pateiktas naujas konceptualus A.L. Rozenbergo teoremos apie žiedo stipriai pirminio radikalo struktūrą įrodymas.