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The sixth power moment of the Riemann zeta-function in the critical strip

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The aim of this paper is to obtain the asymptotics for the quantity

$$\int_1^\infty |\zeta(\sigma+it)|^{2k}\,\mathrm{d}t,$$

when $k=3$ in the region $1/2<\sigma<1$. The cases $k=1,2$ have been completely investigated earlier and explicit asymptotic formulas have been found.

Generally, we have (see [1])

$$\int_1^T |\zeta(\sigma+it)|^{2k}\,\mathrm{d}t = T\sum_{n=1}^{\infty} d_k^2(n)n^{-2\sigma} + R(k,\sigma;T),$$

were $R(k,\sigma;T)=\mathrm{o}(T)$ as $T\to\infty$ and $1/2<\sigma<1$. Some improvements of the estimate for the error term $R(k,\sigma;T)$ were obtained assuming special cases for k in subranges of the critical strip $1/2<\sigma<1$. The latest results can be found in [1]. We will consider the case $k=3$. The best estimate for the error term was obtained recently by A. Ivič [1] and has the form

$$R(3,\sigma;T) \ll_\varepsilon T^{(17-12\sigma)/10}, \quad \frac{7}{12}<\sigma<1.$$

Applying the new approximate functional equation for $|\zeta(\sigma+it)|^{2k}$ and the machinery of the paper [2] we may deduce the estimate for the error term $R(3,\sigma;T)$ valid in the whole critical strip.

Lemma 1. *Let $k\geqslant 1$ be an integer number. There exist constants $c\geqslant 1$, $\alpha_1(u,v)$, an integer U and the polynomials of U-th degree $\alpha_2(u,v)=P_{u,v}(\sigma)$, all depending only on k, such that*

$$|\zeta(\sigma+it)|^{2k} = \sum_{mn\leqslant cT^k} d_k(m)d_k(n)(mn)^{-\sigma}\left(\frac{m}{n}\right)^{it}K_1(mn,t)$$

$$+\left(\frac{t}{2\pi}\right)^{-k(2\sigma-1)}\sum_{mn\leqslant cT^k} d_k(m)d_k(n)(mn)^{\sigma-1}\left(\frac{m}{n}\right)^{it}K_2(mn,t)$$
$$+\,\mathrm{O}(T^{-2})$$

uniformly for $T \leqslant t \leqslant 2T$ *and* $1/2 < \sigma \leqslant \sigma_0 < 1$*, where*

$$K_j(x,t) = \frac{1}{2\pi i}\int\limits_{1-i\infty}^{1+i\infty}\left\{\frac{(t/2\pi)^k}{x}\right\}^z \mathrm{e}^{z^2/T}\left(1+\sum_{u=0}^{U}\sum_{v}\alpha_j(u,v)z^u t^{-2v}\right)\frac{\mathrm{d}z}{z},$$

$j = 1, 2$*, and* $\sum_v$ *denotes the summation over*

$$\mathrm{Max}\left(1, \frac{u}{3}\right) \leqslant v \leqslant u.$$

The proof of this Lemma can be found in [2]. In the cases $k = 1, 2$ this approximate functional equation implies the explicit formulas for the mean values.

Let T_1 and T_2 be any real numbers such that $T \leqslant T_1 \leqslant T_2 \leqslant 2T$. First we consider the integral

$$\int\limits_{T_1}^{T_2} |\zeta(\sigma+it)|^{2k}\,\mathrm{d}t.$$

Substituting the expression of the lemma in the integral we obtain

$$\int\limits_{T_1}^{T_2} |\zeta(\sigma+it)|^{2k} = \sum_{mn\leqslant cT^k} d_k(m)d_k(n)(mn)^{-\sigma}\int\limits_{T_1}^{T_2}\left(\frac{m}{n}\right)^{it}K_1(mn,t)\,\mathrm{d}t$$
$$+\sum_{mn\leqslant cT^k} d_k(m)d_k(n)(mn)^{\sigma-1}\int\limits_{T_1}^{T_2}\left(\frac{m}{n}\right)^{it}\left(\frac{t}{2\pi}\right)^{-2k(\sigma-1/2)}K_2(mn,t)\,\mathrm{d}t + \mathrm{O}(1).$$

The calculation of the above sums is complicated and needs serious machinery. All these general calculations are done in the paper [2] for special case $k = 2$. Applying the same method it is not difficult to obtain the corresponding estimates in the case $k = 3$. Splitting the calculations in the case $u = 0$ and $u \geqslant 1$, and later $m = n$, $m \neq n$, we find that

$$\int\limits_{T_1}^{T_2} |\zeta(\sigma+it)|^{6}\mathrm{d}t = \sum_{m^2\leqslant cT^3} d_3^2(m)(m)^{-2\sigma}\int\limits_{T_1}^{T_2} H(m^2,t)\,\mathrm{d}t$$
$$+\sum_{m^2\leqslant cT^3} d_3^2(m)(m)^{-2+2\sigma}\int\limits_{T_1}^{T_2}\left(\frac{t}{2\pi}\right)^{-6(\sigma-1/2)}H(m^2,t)\,\mathrm{d}t$$

$$+ \sum_{mn\leqslant cT^3, m\neq n} d_3(m)d_3(n)(mn)^{-\sigma}\left(\frac{m}{n}\right)^{it}\left(i\log\frac{m}{n}\right)^{-1}H(mn,t)\Big|_{T_1}^{T_2}$$

$$+ \sum_{mn\leqslant cT^3, m\neq n} d_3(m)d_3(n)(mn)^{\sigma-1}\left(\frac{m}{n}\right)^{it}\left(i\log\frac{m}{n}\right)^{-1}\left(\frac{t}{2\pi}\right)^{-6(\sigma-1/2)}H(mn,t)\Big|_{T_1}^{T_2}$$

$$+ \sum_{mn\leqslant cT^3, m\neq n} d_3(m)d_3(n)(mn)^{\sigma-1}\left(i\log\frac{m}{n}\right)^{-1}\left(\frac{m}{n}\right)^{iaT}H(mn,aT)$$

$$\times\left[\left(\frac{T_2}{2\pi}\right)^{-6(\sigma-1/2)} - \left(\frac{T_1}{2\pi}\right)^{-6(\sigma-1/2)}\right] + O(T^{3(1-\sigma)-1/2+\varepsilon}) + O(T^{\varepsilon}),$$

uniformly in σ, $1/2 < \sigma \leqslant \sigma_0 < 1$ where $H(mn,t) = 0$ or $H(mn,t) = 0$ for $mn > (t/2\pi)^3$ or $mn < (t/2\pi)^3$, respectively, and a, $1 < a < 2$, is a constant.

Now putting $T_2 = 2T_0$, $T_1 = T_0$, $T_0 = T/2^n$ and summing over n, we obtain by [2]

$$\int_0^T |\zeta(\sigma + it)|^6 \,\mathrm{d}t = T \sum_{m\leqslant(T/2\pi)^{3/2}} d_3^2(m)m^{-2\sigma} + O(T^{3(1-\sigma)-1/2+\varepsilon}) + O(T^{\varepsilon}).$$

The last equality can be rewritten in the following form

$$\int_0^T |\zeta(\sigma + it)|^6 \mathrm{d}t = T\sum_{m=1}^{\infty} d_3^2(m)m^{-2\sigma} + O(T^{5/2-3\sigma+\varepsilon}) + O(T^{\varepsilon}),$$

uniformly in σ, $1/2 < \sigma \leqslant \sigma_0 < 1$. For $k > 3$ this method does not give sharper estimates for the function $R(k,\sigma;T)$.

References

[1] A. Ivič, *Higher Moments of the Riemann Zeta-function in the Critical Strip*, Preprint (2001).

[2] А. Каченас, Асимптотик четвертого степенного момент дзет-функции Риман в критическом полосе, *Liet. Matem. Rink.*, **36**(1), 39–54 (1996).

Šešetasis Rymano dzeta funkcijos momentas kritinėje juostoje

A. Kačėnas

Straipsnyje nagrinėjamas šeštasis Rymano dzeta funkcijos momentas kritinėje juostoje. Gautas įvertis

$$\int_0^T |\zeta(\sigma + it)|^6 \mathrm{d}t = T\sum_{m=1}^{\infty} d_3^2(m)m^{-2\sigma} + O(T^{5/2-3\sigma+\varepsilon}) + O(T^{\varepsilon}),$$

tolygiai σ atžvilgiu, kai $1/2 < \sigma \leqslant \sigma_0 < 1$.