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A limit theorem for the Hurwitz zeta-function on the space of meromorphic functions

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Let $s = \sigma + it$ be a complex variable. The Hurwitz zeta-function is defined for $\sigma > 1$ by the following Dirichlet series

$$\zeta(\alpha, s) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s},$$

here $\alpha \in \mathbb{R}$, $0 < \alpha \leqslant 1$, is a fixed parameter. The function is analytically continuable over the complex plane except for a simple pole at the point $s = 1$ with the residue 1.

Let for $N \in \mathbb{N}$

$$\mu_N(\ldots) = \frac{1}{N+1} \#\{0 \leqslant k \leqslant N, \ldots\},$$

here instead of dots we write a condition satisfied by k.

Let $h > 0$ be a fixed number such that $\exp\{2\pi/h\}$ is rational. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. We assume that α is a transcendental number.

Let $d(z_1, z_2)$ be a metric on the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ given by the formulae

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}, \quad d(z, \infty) = \frac{2}{\sqrt{1+|z|^2}}, \quad d(\infty, \infty) = 0,$$

$z, z_1, z_2 \in \mathbb{C}$. This metric is compatible with the topology of $\mathbb{C}_\infty$.

Denote by $M(D)$ the space of meromorfic on D functions $f: D \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta.

Define a probability measure

$$P_N(A) = \mu_N\big(\zeta(\alpha, s + ikh) \in A\big), \quad A \in \mathcal{B}\big(M(D)\big).$$

In this note we present a discrete limit theorem for the Hurwitz zeta-function $\zeta(\alpha, s)$ on the space of meromorphic functions:

Theorem. *There exists a probability measure P on $\Big(M(D), \mathcal{B}\big(M(D)\big)\Big)$ such that the probability measure P_N converges weakly to P as $N \to \infty$.*

Let

$$f_1(s) = 1 - 2^{1-s},$$

and

$$f_2(s) = \zeta(\alpha, s) f_1(s).$$

Then the function $f_2(s)$ is analytic on D. Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta.

Define probability measures

$$P_{N,f_1}(A) = \mu_N\big(f_1(s+ikh) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

and

$$P_{N,f_2}(A) = \mu_N\big(f_2(s+ikh) \in A\big), \quad A \in \mathcal{B}(H(D)).$$

Lemma 1. *There exists a probability measure P_{f_1} on $\big(H(D), \mathcal{B}(H(D))\big)$ such that the probability measure P_{N,f_1} converges weakly to P_{f_1} as $N \to \infty$.*

Proof. The function $f_1(s)$ is a Dirichlet polynomial. Therefore the proof coincides with that of Lemma 1 from [4].

Lemma 2. *There exists a probability measure P_{f_2} on $\big(H(D), \mathcal{B}(H(D))\big)$ such that the probability measure P_{N,f_2} converges weakly to P_{f_2} as $N \to \infty$.*

Reasoning similarly to the proof of Theorem from [4] we obtain the assertion of the lemma.

The functions $f_1(s)$ and $f_2(s)$ have a discrete limit distribution on the space $H(D)$. Now we will prove a joint limit theorem for these functions.

Denote by

$$F(s) = \big(f_1(s), f_2(s)\big)$$

is a $H^2(D)$-valued random element, here $H^2(D)$ denotes the Cartesian product $H^2(D) \times H^2(D)$. Let P_F stand for the distribution of $F(s)$, and define a probability measure

$$P_{N,F}(A) = \mu_N\big(F(s+ikh) \in A\big), \quad A \in \mathcal{B}(H^2(D)).$$

Lemma 3. *There exists a probability measure P_F on $\big(H(D), \mathcal{B}(H(D))\big)$ such that the probability measure $P_{N,F}$ converges weakly to P_F as $N \to \infty$.*

Proof. The first step. By Lemmas 1 and 2 we have that P_{N,f_j} converges weakly to P_{f_j} as $N \to \infty$, $j = 1, 2$. Consequently, the family of probability measures $\{P_{N,f_j}\}$ is tight, $j = 1, 2$. Then for each $\varepsilon > 0$ there exists a compact set $K_j \in H(D)$, $j = 1, 2$, such that

$$P_{N,f_j}\big(H(D) \setminus K_j\big) < \frac{\varepsilon}{2}, \quad j = 1, 2. \tag{3.1}$$

Let θ_N be a random variable on $(\widetilde{\Omega}, \mathcal{F}, \mathbb{P})$ with the distribution

$$\mathbb{P}(\theta_N = kh) = \frac{1}{N+1}, \quad k = 0, 1, \ldots, N.$$

Define a $H^2(D)$-valued random element

$$F_N(s) = \big(f_1(s + i\theta_N), f_2(s + i\theta_N)\big).$$

From the definition of P_{N,f_j} and (3.1), $j = 1, 2$, it follows that

$$P_{N,f_j}\big(H(D) \setminus K_j\big) = \mathbb{P}\big(f_j(s + i\theta_N) \in H(D) \setminus K_j\big) < \frac{\varepsilon}{2}, \quad j = 1, 2.$$

Now let us take $K = K_1 \times K_2$. Then

$$P_{N,F}\big(H^2(D) \setminus K\big) = \mathbb{P}\big(F_N \in H^2(D) \setminus K\big) < \varepsilon.$$

In a such way the family $\{P_{N,F}\}$ is tight.

The second step. Denote by $s_1, \ldots, s_r$ arbitrary points on D, and let the function u: $H^2(D) \to H(D)$ be given by the formula

$$u(f_1, f_2) = \sum_{m=1}^{r} a_{1m} f_1(s_m + s) + \sum_{m=1}^{r} a_{2m} f_2(\alpha, s_m + s),$$

here $a_{1m}, a_{2m} \in \mathbb{C}$, $s \in D$. Set, for $\sigma > \frac{1}{2}$,

$$Z(s) = u\big(f_1(s), f_2(s)\big).$$

Then

$$Z(s) = \sum_{m=1}^{r} a_{1m} f_1(s_m + s) + \sum_{m=1}^{r} a_{2m} f_2(s_m + s).$$

For $\sigma > \frac{1}{2}$ the functions f_1 and f_2 are presented by absolutely convergent Dirichlet series

$$f_1(s) = \sum_{l=1}^{\infty} \frac{b_{1l}}{l^s}, \quad f_2(s) = \sum_{l=1}^{\infty} \frac{b_{2l}}{(l+\alpha)^s}, \quad b_{1l}, b_{2l} \in \mathbb{C}.$$

Consequently,

$$Z(s)=\sum_{l=1}^{\infty}\sum_{m=1}^{r}\frac{a_{1m}b_{1l}}{l^{s_m+s}}+\sum_{l=1}^{\infty}\sum_{m=1}^{r}\frac{a_{2m}b_{2l}}{(l+\alpha)^{s_m+s}}.$$

Repeating the proof of Lemma 2, we obtain that probability measure

$$P_{N,Z}(A)=\mu_N\bigl(Z(s+ikh)\in A\bigr),\quad A\in\mathcal{B}\bigl(H(D)\bigr), \tag{3.2}$$

converges weakly to some probability measure P_Z as $N\to\infty$.

The third step. Since the family $\{P_{N,F}\}$ is relatively compact, we can find a subsequence $\{P_{N',F}\}$ which converges weakly to P^* as $N'\to\infty$. Suppose that P^* is the distribution of some $H^2(D)$-valued random element

$$F^*(s)=\bigl(f_1^*(s),f_2^*(s)\bigr).$$

Therefore

$$F_{N'}\xrightarrow[N'\to\infty]{\mathcal{D}}F^*. \tag{3.3}$$

Taking into account the continuity of the function u we deduce that

$$u(F_{N'})\xrightarrow[N'\to\infty]{\mathcal{D}}u(F^*).$$

Switching to the definition of $Z(s)$ we have

$$Z(s+i\theta_{N'})\xrightarrow[N'\to\infty]{\mathcal{D}}u(F^*).$$

By the second step of the proof

$$Z(s+i\theta_{N'})\xrightarrow[N'\to\infty]{\mathcal{D}}u(F).$$

Consequently,

$$u(F)=u(F^*) \tag{3.4}$$

in the sense of distribution.

Now let $u_1\colon H(D)\to\mathbb{C}$ be defined by the formula

$$u_1(f)=f(0),\quad f\in H(D).$$

Then the function u_1 is a random element, and (3.5) gives

$$u(F)(0)=u(F^*)(0)$$

in the sense of distribution. The definition of u yields

$$\sum_{m=1}^{r} a_{1m} f_1(s_m) + \sum_{m=1}^{r} a_{2m} f_2(s_m) = \sum_{m=1}^{r} a_{1m} f_1^*(s_m) + \sum_{m=1}^{r} a_{2m} f_2^*(s_m)$$

in the sense of distribution with $a_{1m}, a_{2m} \in \mathbb{C}$. The hyperplanes form a determining class on the space $\mathbb{R}^{4n}$ [1]. Therefore, the hyperplanes also form a determining class on the space $\mathbb{C}^{2n}$. From (3.4) it follows that $\mathbb{C}^{2n}$-valued random elements $f_j(s_m)$, $j = 1, 2$, $m = 1, \ldots, r$, have the same distribution as $f_j^*(s_m)$, $j = 1, 2, m = 1, \ldots, r$.

Let K be a compact subset of D, and a sequence $\{s_m\}$ to be relatively compact in K. For $\varepsilon > 0$ we construct the sets

$$S = \Big\{(g_1, g_2) \in H^2(D): \sup_{s \in K} |g_j(s) - f_j(s)| < \varepsilon,\ f_j \in H(D),\ j = 1, 2\Big\},$$

and

$$S_n = \Big\{(g_1, g_2) \in H^2(D): |g_j(s_m) - f_j(s_m)| < \varepsilon,\ f_j \in H(D),\ j = 1, 2,\ m = 1, \ldots, n\Big\}.$$

The same distribution of f_j and f_j^*, $j = 1, 2$, implies that $F^*(\alpha, s) \in S_n$. Since $\{s_m\}$ is relatively compact, then $F^*(\alpha, s) \in S$. It is known that $\{\cap_{n=1}^{m} S_n,\ m \in \mathbb{N}\}$ form a determining class [1]. Hence we obtain

$$F^* = F$$

in the sense of distribution.

This and (3.3) yield

$$F_{N'} \xrightarrow[N' \to \infty]{\mathcal{D}} F.$$

Since the random element F does not depend on the choice of the sequence N', we obtain the assertion of the lemma.

Proof of Theorem

Now we define the function $v: H^2(D) \to M(D)$ by the formula

$$v(f_1, f_2) = \frac{f_1}{f_2}, \quad f_1, f_2 \in H(D).$$

The definition of the metric d on $\mathbb{C}_\infty$ implies

$$d(f_1, f_2) = d\Big(\frac{1}{f_1}, \frac{1}{f_2}\Big).$$

Consequently, the function v is continuous. In view of the definition of v

$$\mu_N\big(\zeta(\alpha, s+ikh)\in A\big)=\mu_N\Big(v\big(f_1(s+ikh), f_2(s+ikh)\big)\in A\Big),\ A\in\mathcal{B}\big(M(D)\big).$$

Applying Lemma 3 we deduce that the probability measure $P_N = P_{N,F}u^{-1}$ converges weakly to some probability measure P as $N \longrightarrow \infty$.

Theorem is proved.

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Ribinė teorema Hurwitz'o dzeta funkcijai meromorfinių funkcijų erdvėje

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Įrodoma diskreti ribinė teorema Hurwitz'o dzeta funkcijai meromorfinių funkcijų erdvėje.