Liet. matem. rink., T. 41, spec. nr., 45–52

A limit theorem for the Riemann zeta-functions in the space of continuous functions

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Let $s = \sigma + it$ be a complex variable, and let, as usual, $\zeta(s)$ denote the Riemann zeta-function. H. Bohr and B. Jessen observed in the third decade of the last century that for the investigation of value-distribution of $\zeta(s)$ probabilistic methods could be applied. They also proved the first limit theorem on the complex plane $\boldsymbol{C}$. Later many mathematicians continued investigations of Bohr-Jessen. Among them A. Wintner, A. Selberg, P.D.T.A. Elliott, A. Ghosh, B. Bagchi, K. Matsumoto, A. Laurinčkas and others. The most of results in the field can be found in the monograph [2]. The aim of this note is to prove a limit theorem for $\zeta(s)$ in the space of continuous functions.

Let $\boldsymbol{C}_\infty = \boldsymbol{C} \cup \infty$ be the Riemann sphere with a metric $d(s_1, s_2)$ given by the formulae

$$d(s_1,s_2) = \frac{2|s_1 - s_2|}{\sqrt{1+|s_1|^2}\sqrt{1+|s_2|^2}}, \quad d(s_1,\infty) = \frac{2}{\sqrt{1+|s_1|^2}}, \quad d(\infty,\infty) = 0,$$

$s_1, s_2 \in \boldsymbol{C}$. Let $C(\boldsymbol{R}) = C(\boldsymbol{R}, \boldsymbol{C}_\infty)$ denote the space of continuous functions $f : \boldsymbol{R} \to \boldsymbol{C}_\infty$ equipped with the topology of uniform convergence on compacta.

Let γ be the unit circle on $\boldsymbol{C}$, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. With the product topology and pointwise multiplication Γ is a compact topological Abelian group. Therefore there exists the probability Haar measure m_H on $(\Gamma, \boldsymbol{B}(\Omega))$, where $\boldsymbol{B}(S)$, stands for the class of Borel sets of the space $\boldsymbol{S}$. This gives a probability space $(\Omega, \boldsymbol{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and put

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p).$$

Moreover, by $d_a(m)$ denote the coefficients of the Dirichlet series expansion of $\zeta^a(s)$ in the half-plane $\sigma > 1$, and define

$$\beta_T(t,\omega) = \sum_{m \leqslant T} \frac{d_{\kappa_T}(m)\omega(m)}{m^{\sigma_T + it}}.$$

Later it will be proved that $\beta_T(t,\omega)$ for almost all $\omega \in \Omega$ converges uniformly in t on compact subsets of $\boldsymbol{R}$ to some function $\beta(t,\omega)$ as $T \to \infty$. Therefore, $\beta(t,\omega)$ is a $C(\boldsymbol{R})$-valued random element defined on $(\Omega, \boldsymbol{B}(\Omega), m_H)$. Denote by P_β its distribution. Let

$$\nu_T(\ldots) = \frac{1}{T}\text{meas}\{\tau \in [0,T], \ldots\}.$$

Theorem. *Let $\theta > \sqrt{2}/2$ be fixed, $\kappa_T = (2^{-1}\log\log T)^{-1/2}$ and $\sigma_T = 1/2 + \theta(\log\log T)^{3/2}(\log T)^{-1}$. Then under the Riemann hypothesis the probability measure*

$$P_T(A) \overset{def}{=} \nu_T\big(\zeta^{\kappa_T}(\sigma_T + it + i\tau) \in A\big), \quad A \in \boldsymbol{B}(C(\boldsymbol{R})),$$

converges weackly to P_β as $T \to \infty$.

First we will prove the existance of the random element $\beta(t,\omega)$. Let

$$Z_{nk}(t,\omega) = \beta_{n+k}(t,\omega) - \beta_n(t,\omega).$$

For any compact subset K of $\boldsymbol{R}$ and every $\varepsilon > 0$ we define the set $A^\varepsilon_{nk}(K)$ by

$$A^\varepsilon_{nk}(K) = \left\{\omega \in \Omega : \sup_{t\in K} |Z_{nk}(t,\omega)| \geqslant \varepsilon\right\},$$

and let

$$A^\varepsilon_k(K) = \bigcap_{l=1}^{\infty} \bigcup_{n>l} A^\varepsilon_{nk}(K)$$

Lemma 1. *We have $m_H(A^\varepsilon_k(K)) = 0$ for every $\varepsilon > 0$, K, and $k \in N$.*

Proof is similar to that of Lemma 7.1.2 from [2].

Lemma 2. *There exists a function $\beta(t,\omega)$ such that for almost all $\omega \in \Omega$ $\beta_T(t,\omega) \to \beta(t,\omega)$, as $T \to \infty$, uniformly in t on compact subsets of $\boldsymbol{R}$.*

Proof. It is not difficult to see that

$$\beta_T(t,\omega) = \sum_{m\leqslant[T]} \frac{d_{\kappa_{[T]}}(m)\omega(m)}{m^{\sigma_{[T]}+it}} + \frac{B}{T^{1/3}} \tag{1}$$

uniformly in $t \in \boldsymbol{R}$ and $\omega \in \Omega$.

Let $\{K_j\}$ be a sequence of compact subsets of $\boldsymbol{R}$ such that $R = \bigcup_{j=1}^{\infty} K_j$, $K_j \subset K_{j+1}$, and if K is a compact of $\boldsymbol{R}$, then $K \subset K_j$ for some j. Let

$$\rho_j(f,g) = \sup_{t\in K_j} d\big(f(t), g(t)\big), \quad f, g \in \boldsymbol{C}(\boldsymbol{R}).$$

Then

$$\rho(f,g)=\sum_{j=1}^{\infty}2^{-j}\frac{\rho_j(f,g)}{1+\rho_j(f,g)}$$

is a metric in $C(\boldsymbol{R})$. However, $C(\boldsymbol{R})$ is a separable metric space. Therefore

$$m_H\big(\omega\in\Omega:\beta_T(t,\omega)\not\to\big)=m_H\Bigg(\omega\in\Omega:\omega\in\bigcup_{\varepsilon>0}\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}A_k^{\varepsilon}(K_j)\Bigg).$$

Hence by Lemma 1 and (1) we have that

$$m_H\big(\omega\in\Omega:\beta_T(t,\omega)\not\to\big)=0.$$

Consequently, there exists a function $\beta(t,\omega)$ such that for almost all $\omega\in\Omega$ the function $\beta_T(t,\omega)$ converges to $\beta(t,\omega)$ uniformly in t on compact subsets of $\boldsymbol{R}$ as $T\to\infty$.

Lemma 2 shows that $\beta(t,\omega)$ is a $C(\boldsymbol{R})$-valued random element defined on the probability space $(\Omega,\boldsymbol{B}(\Omega),m_H)$.

Now let

$$S_u(s)=\sum_{m<u}\frac{d_{\kappa_T}(m)}{m^s},$$

and concider the weak convergence of the probability measure

$$P_{T,S_T}(A)=\nu_T\big(S_T(\sigma_T+it+i\tau)\in A\big),\quad A\in\boldsymbol{B}(C(\boldsymbol{R})).$$

We begin with a limit theorem on the torus Ω. Let

$$Q_T(A)=\nu_T\big((p_1^{-i\tau},p_2^{-i\tau},\ldots)\in A\big),\quad A\in\boldsymbol{B}(\Omega),$$

where p_m stands for the mth prime number.

Lemma 3. *The probability measure Q_T converges weakly to the Haar measure m_H on $(\Omega,\boldsymbol{B}(\Omega))$, as $T\to\infty$.*

Proof is given in [2], Lemma 7.1.1.

Lemma 4. *On $(C(\boldsymbol{R}),\boldsymbol{B}(C(\boldsymbol{R})))$ there exists a probability measure P such that the measure P_{T,S_T} converges weakly to P as $T\to\infty$.*

Proof. Let $h_T:\Omega\to C(\boldsymbol{R})$ be given by the formula

$$h_T(t,\omega)=\sum_{m\leqslant T}\frac{d_{\kappa_T}(m)}{m^{\sigma_T+it}}\prod_{p^\alpha\|m}\omega^\alpha(p)=\beta_T(t,\omega).$$

Then we have that

$$S_T(\sigma_T + it + i\tau) = h_T(t, p_1^{-i\tau}, p_2^{-i\tau}, \ldots). \tag{2}$$

By Lemma 2 we have that for almost all $\omega h_T(t,\omega) \to \beta(t,\omega)$ uniformly in t on compact subsets of $\boldsymbol{R}$.

Now let

$$E = \{\omega \in \Omega : h_T(t, \omega_T) \to \beta(t,\omega) \quad \text{for some} \quad \omega_T \to \omega\}.$$

The space Ω is compact, therefore it is separable. Concequently, $E \in \boldsymbol{B}(\Omega)$. We will prove that $m_H(E) = 0$.

We can write $\omega_T = \omega(e^{i\tau_{\beta_1}(T)}, e^{i\tau_{\beta_2}(T)}, \ldots)$, where $\tau_{\beta_j}(T) \to 0$, as $T \to \infty$, $j = 1, 2, \ldots$. Repeating the proof of Lemmas 1 and 2 we obtain that

$$\sum_{m \leqslant T} \frac{d_{\kappa_T}(m)\omega(m)}{m^{\sigma_T + it}} \prod_{p_j^\alpha || m} e^{i\alpha\tau_{p_j}}, \tag{3}$$

for almost all $\omega \in \Omega$ converges to some function $h(t, \omega; \tau_{p_1}, \tau_{p_2}, \ldots)$ uniformly in t on compact subsets of $\boldsymbol{R}$, uniformly in τ_{p_1} on compact subsets of $\boldsymbol{R}$, uniformly in τ_{p_2} on compact subsets of $\boldsymbol{R}$, ... Hence we obtain that the sequence (3) for almost all $\omega \in \Omega$ converges to $h(t, \omega; \tau_{p_1}, \tau_{p_2}, \ldots)$ as $T \to \infty$ uniformly in $t, \tau_{p_1}, \tau_{p_2}, \ldots$ on compact subsets of $\boldsymbol{R}$. Therefore from this it follows that

$$\sum_{m \leq T} \frac{d_{\kappa_T}(m)\omega(m)}{m^{\sigma_T + it}} \prod_{p_j^\alpha || m} e^{i\alpha\tau_{p_j}(T)}$$

for almost all $\omega \in \Omega$ converges to $h(t, \omega; 0, 0, \ldots)$ uniformly in t on compact subsets of $\boldsymbol{R}$ as $T \to \infty$. Clealy, $h(t, \omega; 0, 0, \ldots) = \beta(t, \omega)$. This means that $m_H(E) = 0$. Therefore, by (2), Lemma 3 and Theorem 5.5 from [1] the lemma follows.

Lemma 5. *The probability measure P_{T,S_T} converges weakly to the measure P_β as $T \to \infty$.*

Proof. The assertion of the lemma was obtained in the proof of Lemma 4. Really, the limit measure in Lemma 4 is $m_H\beta^{-1}$. By the definition of $m_H\beta^{-1}$ this means that

$$P(A) = m_H\beta^{-1}(A) = m_H(\beta^{-1}A) = m_H(\omega \in \Omega : \beta(t,\omega) \in A) = P_\beta(A),$$
$$A \in \boldsymbol{B}(C(\boldsymbol{R})).$$

Now let $n_T = T^{\kappa_T/2}$ and $\varepsilon_T = (\log\log T)^{-1}$.

Lemma 6. *The probability measure $P_{T,S_{n_T}}$ converges weakly to the measure P_β as $T \to \infty$.*

Proof. Let

$$Z_T(t,\tau) = \sum_{n_T<m\leqslant T} \frac{d_{\kappa_T}(m)}{m^{\sigma_T+it+i\tau}}.$$

Then for any compact subset K of $\boldsymbol{R}$ we have

$$\nu_T\big(\sup_{s\in K}|Z_T(t,\tau)| \geqslant \varepsilon_T\big) \leqslant \frac{1}{\varepsilon_T^2 T}\int_0^T \sup t\in K|Z_T(t,\tau)|^2\,\mathrm{d}\tau$$

$$= \frac{B\log T}{\varepsilon_T^2 T}\int_0^T\int_L |Z_T(z,\tau)|^2|\mathrm{d}z|,$$

where L is the contour similar to that in the proof of Lemma 1. Therefore by the Montgomery-Vaughan theorem

$$\nu_T\big(\sup_{s\in K}|Z_T(t,\tau)| \geqslant \varepsilon_T\big) = \frac{B\log T}{\varepsilon_T^2 T}\sup z\in L\int_0^T |Z_T(z,\tau)|^2\mathrm{d}z$$

$$= \frac{B\log T}{\varepsilon_T^2 T}\sum_{n_T<m\leqslant T}\frac{d^2_{\kappa_T}(m)}{m^{2\sigma_T+2u}} = \frac{B\log T}{\varepsilon_T^2}T^{-\kappa_T\frac{\theta(\log\log T)^{3/2}-2}{\log T}}$$

$$\times\sum_{n_T<m\leqslant T}\frac{d^2_{\kappa_T}(m)}{m} = \mathrm{o}(1),$$

as $T\to\infty$. From this and definition of the metric ρ we find that

$$\nu_T\Big(\rho(S_T(\sigma_T+it+i\tau), S_{n_T}(\sigma_T+it+i\tau)) \geqslant \varepsilon\Big) = \mathrm{o}(1), \quad \text{as} \quad T\to\infty.$$

Let

$$g(s) = \zeta^{\kappa_T}(s) - S_{n_T}(s), \quad K(\sigma) = \int_{-\infty}^{\infty} |g(\sigma+it)|^2 w(t)\,\mathrm{d}t,$$

where

$$w(t) = \int_{\log^2 T}^{T/2} \mathrm{e}^{-2(t-2\tau)^2\mathrm{d}\tau}.$$

The Riemann hypothesis is used only in the next lemma and Lemma 11.

Lemma 7. *Let* $1/2 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant 9/16$ *and* $T \geqslant T_0$. *Then*

$$K(\sigma_2) = B(K(\sigma_1))^{\frac{7-4\sigma_2-4\sigma_1}{7-8\sigma_2}} (T^{1-c_1\kappa_T})^{\frac{4(\sigma_2-\sigma_1)}{7-8\sigma_1}} + B(K(\sigma_1))^{\frac{7-8\sigma_2}{7-8\sigma_1}} e^{-c_2(\sigma_2-\sigma_1)\log^4 T}$$

with positive constants c_1 *and* c_2.

The lemma is Lemma 7.2.2 from [2].
Now let

$$L(\sigma) = \int_{-\infty}^{\infty} |S_{n_T}(\sigma+it)|^{2/\kappa_T} w(t)\,\mathrm{d}t, \quad J(\sigma) = \int_{-\infty}^{\infty} |\zeta(\sigma+it)|^2 w(t)\,\mathrm{d}t.$$

Lemma 8. *Let* $T \geqslant T_0$. *Then the estimate*

$$L\Big(\frac{1}{2}+\frac{1}{\log T}\Big) = BT\log T$$

is valid.

The lemma is a partial case of Lemma 7.2.3. from [2].

Lemma 9. *Let* $T \geqslant T_0$. *Then the estimate*

$$J\Big(\frac{1}{2}+\frac{1}{\log T}\Big) = BT\log T$$

is valid.

The lemma is lemma 7.2.4 from [2] with $l_T = \log T$.

Lemma 10. *Let* $T \geqslant T_0$, *and* $\sigma_T - \frac{1}{\log T} \leqslant \tilde{\sigma}_T \leqslant \sigma_T + \frac{1}{\log T}$. *Then*

$$K(\tilde{\sigma}_T) = BT\exp\big(-c_4(\log\log T)^{3/2}\big).$$

Proof. We take $\sigma_1 = \frac{1}{2}+\frac{1}{\log T}$ and $\sigma_2 \leqslant \sigma_T$ in Lema 7. Clearly,

$$\big|\zeta^{\kappa_T}(s) - S_{n_T}(s)\big| = \Big(|\zeta(s)|^{\kappa_T} + |S_{n_T}(s)|^{(\frac{1}{\kappa_T})\kappa_T}\Big)^2$$
$$\leqslant 2|\zeta(s)|^{2\kappa_T} + 2\Big(|S_{n_T}(s)|^{1/\kappa_T}\Big)^{2\kappa_T} \leqslant \max\Big(4, 2|\zeta(s)|^2 + 2|S_{n_T}(s)|^{2/\kappa_T}\Big).$$

Hence we obtain

$$K(\sigma) = BT + BJ(\sigma) + BL(\sigma),$$

and by Lemas 8 and 9 we have

$$K\left(\frac{1}{2}+\frac{1}{\log T}\right) = BT\log T.$$

This and lemma 7 now yield the estimate of the lemma.

Lemma 11. *Let* $\varepsilon_T = (\log T)^{-1}$ *and let* K *be a compact subset of* $\boldsymbol{R}$. *Then*

$$\nu_T\big(\sup_{s\in K}|g(\sigma_T+it+i\tau)| \geqslant \varepsilon_T\big) = \mathrm{o}(1) \quad as \quad T\to\infty.$$

Proof. By the Chebyshev inequality we have

$$\nu_T\big(\sup_{t\in K}|g(\sigma_T+it+i\tau)| \geqslant \varepsilon_T\big) \leqslant \frac{1}{\varepsilon_T^2 T}\int_0^T \sup_{t\in K}|g(\sigma_T+it+i\tau)|^2\mathrm{d}\tau. \tag{4}$$

Since

$$g^2(\sigma_T+it+i\tau) = \frac{1}{2\pi i}\int_L \frac{g^2(\sigma_T+it+i\tau)}{z-it}\,\mathrm{d}z,$$

where L denotes a simple closed contour enclosing the set iK, we find that

$$\sup_{t\in K}|g(\sigma_T+it+i\tau)|^2 = B\delta^{-1}\int_L |g(\sigma_T+z+i\tau)|^2|\mathrm{d}z|.$$

Here σ is the distance of L from iK. For sufficiantly large T, hence we deduce

$$\int_0^T \sup_{t\in K}|g(\sigma_T+it+i\tau)|^2\mathrm{d}\tau = B\delta^{-1}\int_L |\mathrm{d}z|\int_0^{2T}|g(\sigma_T+\Re z+i\tau)|^2\mathrm{d}\tau.$$

Taking $\delta = (\log T)^{-1}$, in view of Lemma 10, we have

$$\int_0^T \sup_{t\in K}|g(\sigma_T+it+i\tau)|^2\mathrm{d}\tau = B\log T\sup_{z\in L}\int_0^{2T}|g(\sigma_T+\Re z+it)|^2\mathrm{d}t$$

$$= B\log T\int_0^{2T}|g(\tilde{\sigma}_T+it)|^2\mathrm{d}t = BT\exp\big(-c_5(\log\log T)^{3/2}\big),$$

since $\sigma_T - \frac{1}{\log T} \leqslant \tilde{\sigma}_T \leqslant \sigma_T + \frac{1}{\log T}$. From this and from (4) we obtain the lemma.

Proof of the Theorem. For any $\varepsilon > 0$ we have

$$\nu_T\Big(\rho\big(\zeta^{\kappa_T}(\sigma_T + it + i\tau), S_{n_T}(\sigma_T + it + i\tau)\big) \geqslant \varepsilon\Big)$$

$$\leqslant \frac{1}{\varepsilon}\sum_{j=1}^{\infty} 2^{-j}\frac{1}{T}\int_0^T \sup_{t\in K_j}\big|g(\sigma_T + it + i\tau)\big|1 + 2\sup_{t\in K_j}\big|g(\sigma_T + it + i\tau)\big|\,\mathrm{d}\tau$$

$$= \frac{1}{2}\sum_{j=1}^{\infty} 2^{-j}\frac{1}{T}\Bigg(\int\limits_{\substack{0\\ \sup_{t\in K_j}|g(\sigma_T+it+i\tau)|<\varepsilon_T}}^{T} + \int\limits_{\substack{0\\ \sup_{t\in K_j}|g(\sigma_T+it+i\tau)|\geqslant\varepsilon_T}}^{T}\Bigg)$$

$$\times \frac{2\sup_{t\in K_j}\big|g(\sigma_T + it + i\tau)\big|}{1 + 2\sup_{t\in K_j}\big|g(\sigma_T + it + i\tau)\big|}\,\mathrm{d}\tau = \mathrm{o}(1),$$

as $T \to \infty$ by Lemma 11. Therefore the theorem follows from Lemmas 6 and Theorem 4.2 of [1].

References

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York (1968).
[2] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht (1996).

Ribinė teorema Rymano dzeta funkcijai tolydžių funkcijų erdvėje

I. Belov

Straipsnyje įrodoma ribinė teorema tolydžių funkcijų erdvėje silpno tikimybinių matų konvergavimo prasme Rymano dzeta funkcijai arti kritinės tiesės.