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Large deviation theorems for weighted summation of Gamma-distribution random variables

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1. Formulation of the result

Among the problems of limit theorems of large deviations play an important part in the case where separate summands of the sum of random variables satisfy the Cramer or Linnik conditions. Many of the basic ideas and results have been presented fairly completely in the books [1]–[3].

Let $\{X_t, t = 1,2,\ldots\}$ be a sequence of independent random variables (r.v.) with distribution density functions $p_{X_i}(x) = (\lambda^{\alpha_i}/\Gamma(\alpha_i))x^{\alpha_i-1}\exp\{-\lambda x\}$ if $x > 0$ and $p_{X_i}(x) = 0$, $x \leqslant 0$, $i = 1,2,\ldots$. In short, $X_i \sim G(\alpha_i,\lambda)$, where $\alpha_i > 0, \lambda > 0$ and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x$ (the Euler gamma function). It is known that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(n) = (n-1)!$ if $n \in N$. The characteristic function (ch.f.) of the r.v. $X_j \sim G(\alpha_j,\lambda)$, is $f_{X_j}(t) = \mathbf{E}\,\mathrm{e}^{itX_j} = \left(1 - it/\lambda\right)^{-\alpha_j}$, $j = 1,2,\ldots$. Then, the mean and variance of the r.v. X_j are equal to: $\mathbf{E}X_j = \alpha_j/\lambda, \mathbf{D}X_j = \alpha_j/\lambda^2$, respectively. Next, let $\mu_j, j = 1,2,\ldots$, be a sequence of nonrandom real numbers. Denote

$$S_n = \sum_{j=1}^n \mu_j X_j, \quad \mathbf{E}S_n = \frac{1}{\lambda}\sum_{j=1}^n \alpha_j\mu_j, \quad B_n^2 = \mathbf{D}S_n = \frac{1}{\lambda^2}\sum_{j=1}^n \alpha_j\mu_j^2,$$

$$Z_n = B_n^{-1}(S_n - \mathbf{E}S_n), \quad F_n(x) = \mathbf{P}(Z_n < x), \quad p_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}F_n(x), \tag{1}$$

$$\Phi(x) = \int_{-\infty}^{x} \varphi(y)\mathrm{d}y, \quad \varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}x^2\right\}.$$

The targed of our work is to obtain large deviaton theorems and exponential inequalities for the functions $\mathbf{P}(Z_n \geqslant x)$ and $p_n(x)$. First we have to get the estimate of the kth-order cumulant $\Gamma_k(Z_n)$ of the r.v. Z_n, defined by equality (1), where $\Gamma_k(X) := \frac{1}{i^k}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\ln f_X(t)\big|_{t=0}$, $k = 1,2,\ldots$. Here $f_X(t) = \mathbf{E}\exp\{itX\}$ is the ch.f. of the r.v. X.

PROPOSITION. For the kth-order cumulants $\Gamma_k(Z_n)$ of the r.v. Z_n, the estimate

$$\left|\Gamma_k(Z_n)\right| \leqslant (k-1)!\Delta_n^{*\,2-k}, \quad \Delta_n^* = \left(\max_{1\leqslant j\leqslant n}|\mu_j|\right)^{-1}\lambda B_n, \quad k = 3,4,\ldots. \tag{2}$$

Proof. Let $Y_j := \mu_j X_j$, $j = 1, 2, \ldots, n$. Then we have

$$f_{Y_j}(t) = f_{X_j}(\mu_j t) = \left(1 - i\mu_j t/\lambda\right)^{-\alpha_j}. \tag{3}$$

On the basis of definition of the kth-order cumulant and by means of mathematical induction we obtain $\Gamma_k(Y_j) = (k-1)!\,\alpha_j\left(\mu_j/\lambda\right)^k$, $k = 1, 2, \ldots$. Considering that random variables Y_j are independent, we have that $\Gamma_k(S_n) = \sum_{j=1}^n \Gamma_k(Y_j) = (k-1)!\sum_{j=1}^n \alpha_j\left(\mu_j/\lambda\right)^k$, $k = 1, 2, \ldots$. Taking into account that $\Gamma_1(S_n - \mathbf{E}S_n) = 0$, $\Gamma_k(S_n - \mathbf{E}S_n) = \Gamma_k(S_n)$, $k = 2, 3, \ldots$, we get

$$\Gamma_k(Z_n) = B_n^{-k}\Gamma_k(S_n) = (k-1)!\Big(\sum_{j=1}^n \alpha_j\mu_j^2\Big)^{-\frac{k}{2}} \sum_{j=1}^n \alpha_j\mu_j^k, \quad k = 2, 3, \ldots. \tag{4}$$

Hence follows the assertion of the proposition.

Next, let θ_i, $i = 1, 2, \ldots$, stand for quantities not exceeding a unit in absolute value and

$$\Delta_n = c_0\Delta_n^*, \quad c_0 = (1/6)(\sqrt{2}/6), \quad T_n = (1/12)(1 - x/\Delta_n)\Delta_n. \tag{5}$$

Theorem 1. *For the distribution function $F_n(x)$ of the r.v. Z_n defined by equality* (1) *in the interval $0 \leqslant x < \Delta_n$, the relations of large deviations*

$$\begin{aligned} \frac{1 - F_n(x)}{1 - \Phi(x)} &= \exp\{L_n(x)\}\Big(1 + \theta_1 f(x)\frac{x+1}{\Delta_n}\Big), \\ \frac{F_n(-x)}{\Phi(-x)} &= \exp\{L_n(x)\}\Big(1 + \theta_2 f(x)\frac{x+1}{\Delta_n}\Big) \end{aligned} \tag{6}$$

are valid. Here $f(x) = 60\left(1 + 10\Delta_n\exp\{-(1 - x/\Delta_n)\sqrt{\Delta_n}\}\right)\left(1 - x/\Delta_n\right)^{-1}$ and $L_n(x) = \sum_{k=3}^\infty \lambda_{k,n}x^k$. The coefficients $\lambda_{k,n}$ (expressed by cumulants of the r.v. Z_n) are found by formula (2.9) [3]. *In particular,*

$$\begin{aligned} &\lambda_{3,n} = (1/3)\Gamma_3(Z_n), \quad \lambda_{4,n} = (1/24)\left(\Gamma_4(Z_n) - 3\Gamma_3^2(Z_n)\right), \\ &\lambda_{5,n} = (1/120)\left(\Gamma_5(Z_n) - 10\Gamma_3(Z_n)\Gamma_4(Z_n) + 15\Gamma_3^3(Z_n)\right), \ldots. \end{aligned}$$

COROLLARY 1. For $x \geqslant 0$, $x = o\left(\Delta_n^{1/3}\right)$ as $\Delta_n \to \infty$, where Δ_n is determined by (5)

$$\lim_{n\to\infty} \frac{1 - F_n(x)}{1 - \Phi(x)} = 1, \qquad \lim_{n\to\infty} \frac{F_n(-x)}{\Phi(-x)} = 1. \tag{7}$$

COROLLARY 2. For the distribution function $F_n(x)$ of the r.v. Z_n defined by equality (1), the inequality $\sup_x \left|F_n(x) - \Phi(x)\right| \leqslant 18/\Delta_n$ holds.

Theorem 2. *For the r.v. Z_n defined by equality* (1)

$$\mathbf{P}\{\pm Z_n \geqslant x\} \leqslant \begin{cases} \exp\{-\frac{1}{4}x^2\}, & 0 \leqslant x < \Delta_n, \\ \exp\{-\frac{1}{4}\Delta_n x\}, & x \geqslant \Delta_n, \end{cases} \tag{8}$$

here Δ_n is determined by (5).

Theorem 3. *If $\alpha_{i_k} \geqslant \frac{1}{2}$, $k = 1, 2, 3, 4$, where $1 \leqslant i_k \leqslant n$, then for the distribution density function $p_n(x)$ of the r.v. Z_n in the interval $0 \leqslant x < \Delta_n$, for integer l, $l \geqslant 1$, the equality*

$$\frac{p_n(x)}{\varphi(x)} = \exp\{L_n(x)\}\Big(1 + \sum_{\nu=0}^{l-1} M_{\nu,n}(x) + \theta_3 q(l)\Big(\frac{x+1}{\Delta_n}\Big)^l$$
$$+\theta_4 2\pi e^2 \lambda B_n \Big(\prod_{k=1}^{4} |\mu_{i_k}|^{1/4}\Big)^{-1} \exp\big\{-(1/8)T_n^2\big\}\Big) \tag{9}$$

holds. For polynomials $M_{\nu,n}(x)$ the following formula

$$M_{\nu,n}(x) = \sum_{k=0}^{\nu} K_{k,n}(x) q_{\nu-k,n}(x) \tag{10}$$

holds, where $K_{\nu,n}(x) = \sum \prod_{m=1}^{\nu} (k_m!)^{-1}\big(-\lambda_{m+2,n} x^{m+2}\big)^{k_m}$, $K_0(x) = 0$ and $q_{\nu,n}(x) = \sum H_{\nu+2l}(x) \prod_{m+1}^{\nu} (k_m!)^{-1} \big(\Gamma_{m+2}(Z_n)/(m+2)!\big)^{k_m}$, $q_{0,n}(x) \equiv 1$, $H_l(x)$ are Chebyshev–Hermite polynomials, and the summation is taken over all integer solutions of the equation $k_1 + k_2 + \ldots + \nu k_\nu = \nu$. In a special case,

$$M_{0,n}(x) \equiv 0, \quad M_{1,n}(x) = -\frac{1}{2}\Gamma_3(Z_n)x,$$
$$M_{2,n}(x) = \frac{1}{8}(5\Gamma_3^2(Z_n) - 2\Gamma_4(Z_n))x^2 + \frac{1}{24}(3\Gamma_4(Z_n) - 5\Gamma_3^2(Z_n)), \ldots.$$

We get the exspression of the quantity $q(l)$ from (6.7) [3] *that $\gamma = 0$: $q(l) =$* $(3\sqrt{2e}/2)^l 8(l+2)^2 4^{3(l+1)} \Gamma((3l+1)/2)$.

2. Proofs of theorems

Proof of Theorem 3. Since for the kth-order $\Gamma_k(Z_n)$, $k = 2, 3, \ldots$, of the r.v. Z_n, estimate (2) holds, for the r.v. $\xi = Z_n$ the condition (S_γ) with $\gamma = 0$ and $\Delta = \Delta_n$, Δ_n being defined by equality (5) of Lemma (6.1) [1] is satisfied. Basing on this lemma we have to estimate the integral

$$R_n = \int\limits_{|t|\geqslant T_n} \big|f_{Z_n(h)}(t)\big|\mathrm{d}t, \tag{11}$$

where $Z_n(h) = B_n^{-1}(h)\big(S_n(h) - M_n(h)\big)$, $S_n(h) = \sum_{j=1}^n Y_j(h)$ and $Y_j(h)$ is conjugate $Y_j := \mu_j Y_j^2$, $j = 1,2,\dots,n$, r.v. with the density function $p_{Y_j(h)}(x) = \mathrm{e}^{hx}p_{Y_j}(x)\Big(\int_{-\infty}^{\infty}\mathrm{e}^{hx}p_{Y_j}(x)\mathrm{d}x\Big)^{-1}$ and $M_n(h) = \mathbf{E}S_n(h)$, $B_n^2(h) = \mathbf{D}S_n(h)$, $f_{Z_n(h)}(t) = \mathbf{E}\exp\{itZ_n(h)\}$. Further let $\varphi_{Y_j}(h) = \mathbf{E}\,\mathrm{e}^{hY_j} = \int_{-\infty}^{\infty}\mathrm{e}^{hx}p_{Y_j}(x)\mathrm{d}x$. Since $f_{Y_j}(t) = \mathbf{E}\exp\{itY_j\} = \varphi_{Y_j}(it)$, taking into account the exspression of $f_{Y_j}(t)$ by equality (3), we obtain $\varphi_{Y_j}(h) = (1-\mu_j h/\lambda)^{-\alpha_j}$, $j = 1,2,\dots,n$. Hence, basing on the the expression of the density $p_{Y_j(h)}(x)$ of r.v. $Y_j(h)$, we get

$$f_{Y_j(h)}(t) = \big(\varphi_{Y_j}(h)\big)^{-1}\varphi_{Y_j}(h+it) = \big(1-\nu_j(h)it\big)^{-\alpha_j}, \quad \nu_j(h) = \mu_j/(\lambda-\mu_j h), \tag{12}$$

$j = 1,2,\dots,n$. Recaling that Y_j, $j = 1,2,\dots,n$, are independent random variables we obtain

$$f_{Z_n(h)}(t) = \exp\Big\{-it\frac{M_n(h)}{B_n(h)}\Big\}\prod_{j=1}^{n} f_{Y_j(h)}\big(t/B_n(h)\big). \tag{13}$$

Note that $M_n(h) = \mathbf{E}S_n(h) = \sum_{j=1}^n \alpha_j\nu_j(h)$, $B_n^2(h) = \mathbf{D}S_n(h) = \sum_{j=1}^n \alpha_j\nu_j^2(h)$, where $\nu_j(h)$ is defined by equality (12). From this, basing by equality (12), we derive $\big|f_{Z_n(h)}(t)\big| = \prod_{j=1}^n\Big(1+\big(\nu_j^2(h)/B_n^2(h)\big)t^2\Big)^{-\frac{1}{2}\alpha_j}$. Next, using equalities (11) we have

$$R_n = \int\limits_{|t|\geqslant T_n}\exp\Big\{-\frac{1}{2}\sum_{\substack{j=1\\ j\neq i_k}}^{n}\alpha_j\ln\Big(1+\frac{\nu_j^2(h)}{B_n^2(h)}t^2\Big)\Big\}\prod_{k=1}^{4}\big|f_{Y_{i_k}(h)}\big(t/B_n(h)\big)\big|\mathrm{d}t. \tag{14}$$

It is easy to check that

$$\prod_{k=1}^{2}\Big(1+\big(\nu_{i_k}^2(h)/B_n^2(h)\big)t^2\Big)^{-\frac{1}{2}\alpha_{i_k}} \leqslant \Big(1+\big(|\nu_{i_1}(h)\nu_{i_2}(h)|/B_n^2(h)\big)t^2\Big)^{-(\alpha_{i_1}\wedge\alpha_{i_2})}.$$

Then $\int_{-\infty}^{\infty}\prod_{k=1}^{2}\big|f_{Y_{i_k}(h)}\big(t/B_n(h)\big)\big|^2\mathrm{d}t \leqslant \big(\pi B_n(h)/\sqrt{\nu_{i_1}(h)\nu_{i_2}(h)}\big)^{1/2}$ if $\alpha_{i_1}\wedge\alpha_{i_2}\geqslant\frac{1}{2}$. Hence, making use of the Chauchy–Schwarz inequality we obtain

$$\int\limits_{-\infty}^{\infty}\prod_{k=1}^{4}\big|f_{Y_{i_k}(h)}\big(t/B_n(h)\big)\big|\mathrm{d}t \leqslant \pi B_n(h)\Big(\prod_{k=1}^{4}|\nu_{i_k}(h)|^{1/4}\Big)^{-1}. \tag{15}$$

Now, recaling the definition of $\nu_j(h)$ by equality (7) we get $\nu_j(h) = (\mu_j/\lambda)(1 + \theta(1/8))$ and $B_n^2(h) = \mathbf{D}S_n(h) = \sum_{j=1}^n \alpha_j \nu_j^2(h) = \lambda^{-2}\sum_{j=1}^n \alpha_j \mu_j^2(1 + \theta(1/3)) = B_n^2(1 + \theta(1/3))$ for $0 \leqslant h \leqslant \bigl(9\max_{1\leqslant j\leqslant n}|\mu_j|\bigr)^{-1}\lambda$. Now we can easily check that $0 < B_n^{-2}(h)\bigl(\nu_j(h)T_n\bigr)^2 < 1$. Thus, basing on the inequality $\ln(1+x) > (1/2)x$, $0 < x < 1$, we have $\ln\Bigl(1 + \bigl(\nu_j(h)T_n/B_n(h)\bigr)^2\Bigr) \geqslant \bigl(\mu_j T_n/(2\lambda B_n)\bigr)^2$. Hence, taking into account equalities (14) and (15), we obtain the estimate of integral R_n: $R_n \leqslant 2\pi e^2 \lambda B_n \bigl(\prod_{k=1}^4 |\mu_{i_k}|^{1/4}\bigr)^{-1} \exp\{-(1/8)T_n^2\}$, where T_n is defined by equality (5).

To prove Theorem 1, we have to use Lemma 2.3 in [3] for the r.v. $\xi = Z_n$, for the kth-order cumulant $\Gamma_k(Z_n)$, $k = 2, 3, \ldots$, of which estimate (2) holds. To prove Theorem 2, we have to make use of Lemma 2.4 in [3] for the r.v. $\xi = Z_n$, for the kth-order cumulant $\Gamma_k(Z_n), k = 2, 3, \ldots$, of which estimate (2) is valid, considering that $(k-1)! \leqslant (1/2), k = 2, 3, \ldots$.

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Gama-pasiskirsčiusių atsitiktinių dydžių sumavimo su svoriais didžiųjų nuokrypių teoremos

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Darbe gauti Gama-pasiskirsčiusių atsitiktinių dydžių sumavimo su svoriais pasiskirstymo ir jo tankio funkcijų aproksimacijos normaliuoju dėsniu tikslūs įverčiai, atsižvelgiant į asimptotinius skleidinius didžiųjų nuokrypių Kramerio zonoje.