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Convergence of products of independent random variables to the log-Poisson law

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Let us have a set of independent random values $\{\xi_{nk}\}$ in each sequence, where $n \in \mathbb{N}$, $k = 1, 2, 3, \ldots, k_n$ and $k_n \to \infty$, as $n \to \infty$. Suppose that the distribution of ξ_{nk} is defined:

ξ_{nk}	$-\mathrm{e}$	-1	0	1	e
P	p_{nk}^{-}	q_{nk}^{-}	q_{nk}^{0}	q_{nk}^{+}	p_{nk}^{+}

Let the log-Poisson distribution be $\Pi_{\lambda,\mu,\alpha_0,\alpha_1}(u)$ with parameters $\lambda \geqslant 0$, $\mu \geqslant 0$, $\lambda + \mu > 0$, $0 < \alpha_0 \leqslant 1$, $|\alpha_1| \leqslant \alpha_0 \mathrm{e}^{-2\mu}$.

Theorem. *The M-weak limit of random functions* $P(\prod_{k=1}^{k_n} \xi_{nk} < u)$ *is* $\Pi_{\lambda,\mu,\alpha_0,\alpha_1}(u)$ *if and only if the next five conditions are satisfied:*

$$\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} \frac{\mathrm{p}_{nk}^{+} + \mathrm{p}_{nk}^{-}}{1 - \mathrm{q}_{nk}^{0}} = 0, \tag{1}$$

$$\lim_{n\to\infty} \prod_{k=1}^{k_n} (1 - \mathrm{q}_{nk}^{0}) = \alpha_0, \tag{2}$$

$$\lim_{n\to\infty} \prod_{k=1}^{k_n} \left(\mathrm{p}_{nk}^{+} + \mathrm{q}_{nk}^{+} - (\mathrm{p}_{nk}^{-} + \mathrm{q}_{nk}^{-})\right) = \alpha_1, \tag{3}$$

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \frac{\mathrm{p}_{nk}^{+} + \mathrm{p}_{nk}^{-}}{1 - \mathrm{q}_{nk}^{0}} = \lambda + \mu, \tag{4}$$

$$\alpha_1 \left(\lim_{n\to\infty} \sum_{k=1}^{k_n} \frac{\mathrm{q}_{nk}^{+}\mathrm{p}_{nk}^{-} - \mathrm{q}_{nk}^{-}\mathrm{p}_{nk}^{+}}{(\mathrm{q}_{nk}^{+} + \mathrm{p}_{nk})^2 - (\mathrm{q}_{nk}^{-} + \mathrm{p}_{nk}^{-})^2} - \mu \right) = 0. \tag{5}$$

Proof. Sufficiency. Let use Mellin's transformations. Because random variables ξ_{nk} are independent in each sequence, we have:

$$\omega_n^0(t) = \omega^0_{\prod_{k=1}^{k_n} \xi_{nk}}(t) = \prod_{k=1}^{k_n} \left(\mathrm{q}_{nk}^{+} + \mathrm{q}_{nk}^{-} + (\mathrm{p}_{nk}^{+} + \mathrm{p}_{nk}^{-})\mathrm{e}^{it}\right),$$

$$\omega_n^1(t) = \omega^1_{\prod_{k=1}^{k_n}\xi_{nk}}(t) = \prod_{k=1}^{k_n}\left(\mathrm{q}_{nk}^+ - \mathrm{q}_{nk}^- + (\mathrm{p}_{nk}^+ - \mathrm{p}_{nk}^-)\mathrm{e}^{it}\right),$$

$$W_n(t) = \begin{pmatrix} \omega_n^0(t) & 0 \\ 0 & \omega_n^1(t) \end{pmatrix}.$$

The Mellin's transformation of the log-Poisson distribution $\Pi_{\lambda,\mu,\alpha_0,\alpha_1}$ is

$$W(t) = \begin{pmatrix} \alpha_0 \mathrm{e}^{(\lambda+\mu)(\mathrm{e}^{it}-1)} & 0 \\ 0 & \alpha_1 \mathrm{e}^{(\lambda-\mu)(\mathrm{e}^{it}-1)} \end{pmatrix}.$$

Using common theory about M-weak limit of sequence of M-decreasing independent random variables (see, e.g., [2], thm. 6.1), we get that $W_n(t) \to W(t)$ for every $t \in \mathbb{R}$. Hence

$$P\left(\prod_{k=1}^{k_n} \xi_{nk} < u\right) \overset{M}{\Longrightarrow} \Pi_{\lambda,\mu,\alpha_0,\alpha_1}(u),$$

and the sufficiency is proved.

Necessity. Let we have $P(\prod_{k=1}^{k_n} \xi_{nk} < u) \overset{M}{\Longrightarrow} \Pi_{\lambda,\mu,\alpha_0,\alpha_1}(u)$. Hence $W_n(t) \to W(t)$. When $t = 0$, we obtain $\omega_n^0(t) \to \alpha_0, \omega_n^1(t) \to \alpha_1$.

Conditions (2) and (3) are satisfied. On the other hand:

$$\lim_{n\to\infty} \prod_{k=1}^{k_n} \left(\frac{\mathrm{q}_{nk}^+ + \mathrm{q}_{nk}^-}{1-\mathrm{q}_{nk}^0} + \frac{\mathrm{p}_{nk}^+ + \mathrm{p}_{nk}^-}{1-\mathrm{q}_{nk}^0}\mathrm{e}^{it} \right) = \mathrm{e}^{(\lambda+\mu)(\mathrm{e}^{it}-1)} \tag{6}$$

for all $t \in \mathbb{R}$.

Let $\{\eta_{nk}\}$, $n \in \mathbf{N}$, $k = 1, 2, 3, \ldots, k_n$ and $k_n \to \infty$, when $n \to \infty$ is a set of independent random variables in each sequence. Let us suppose that η_{nk} obtains values 0 or 1 with probabilities $\dfrac{\mathrm{q}_{nk}^+ + \mathrm{q}_{nk}^-}{1-\mathrm{q}_{nk}^0}$ and $\dfrac{\mathrm{p}_{nk}^+ + \mathrm{p}_{nk}^-}{1-\mathrm{q}_{nk}^0}$ respectively.

The characteristic function of random variable $\sum_{k=1}^{k_n} \eta_{nk}$ is

$$W_n^*(t) = \prod_{k=1}^{k_n} \left(\frac{\mathrm{q}_{nk}^+ + \mathrm{q}_{nk}^-}{1-\mathrm{q}_{nk}^0} + \frac{\mathrm{p}_{nk}^+ + \mathrm{p}_{nk}^-}{1-\mathrm{q}_{nk}^0}\mathrm{e}^{it} \right).$$

We have $\lim_{n\to\infty} W_n^*(t) = \mathrm{e}^{(\lambda+\mu)(\mathrm{e}^{it}-1)}$ by (6) for all $t \in \mathbb{R}$.

So we can say that

$$\lim_{n\to\infty} P\left(\sum_{k=1}^{k_n} \eta_{nk} = l\right) = \frac{(\lambda+\mu)^l}{l!} \text{ for all } l = 0, 1, 2, \ldots.$$

Hence

$$\begin{cases} \lim\limits_{n\to\infty} P\left(\sum\limits_{k=1}^{k_n} \eta_{nk} = 0\right) = \mathrm{e}^{-(\lambda+\mu)}, \\ \lim\limits_{n\to\infty} P\left(\sum\limits_{k=1}^{k_n} \eta_{nk} = 1\right) = (\lambda+\mu)\mathrm{e}^{-(\lambda+\mu)}, \\ \lim\limits_{n\to\infty} P\left(\sum\limits_{k=1}^{k_n} \eta_{nk} = 2\right) = \dfrac{(\lambda+\mu)^2}{2!}\mathrm{e}^{-(\lambda+\mu)}. \end{cases} \tag{7}$$

On the other hand,

$$\begin{cases} P\left(\sum\limits_{k=1}^{k_n} \eta_{nk} = 0\right) = \prod\limits_{k=1}^{k_n} \dfrac{q_{nk}^{+} + q_{nk}^{-}}{1 - q_{nk}^{0}}, \\ P\left(\sum\limits_{k=1}^{k_n} \eta_{nk} = 1\right) = \sum\limits_{k=1}^{k_n} \dfrac{p_{nk}^{+} + p_{nk}^{-}}{1 - q_{nk}^{0}} \prod\limits_{\substack{k'=1 \\ k'\neq k}}^{k_n} \dfrac{q_{nk'}^{+} + q_{nk'}^{-}}{1 - q_{nk'}^{0}}, \\ P\left(\sum\limits_{k=1}^{k_n} \eta_{nk} = 2\right) = \sum\limits_{k'=1}^{k_n} \sum\limits_{\substack{k''=1 \\ k''>k'}} \dfrac{p_{nk'}^{+}+p_{nk'}^{-}}{1-q_{nk'}^{0}} \dfrac{p_{nk''}^{+}+p_{nk''}^{-}}{1-q_{nk''}^{0}} \prod\limits_{\substack{k=1 \\ k\neq k',k''}}^{k_n} \dfrac{q_{nk}^{+}+q_{nk}^{-}}{1-q_{nk}^{0}}. \end{cases} \tag{8}$$

Because

$$\lim_{n\to\infty}\left(\left(\frac{P(\sum_{k=1}^{k_n}\eta_{nk}=1)}{P(\sum_{k=1}^{k_n}\eta_{nk}=0)}\right)^2 - 2\frac{P(\sum_{k=1}^{k_n}\eta_{nk}=2)}{P(\sum_{k=1}^{k_n}\eta_{nk}=0)}\right) = 0,$$

we get

$$\lim_{n\to\infty}\left(\left(\sum_{k=1}^{k_n}\frac{p_{nk}^{+}+p_{nk}^{-}}{1-q_{nk}^{0}}\,\frac{1-q_{nk}^{0}}{q_{nk}^{+}+q_{nk}^{-}}\right)^2 - \sum_{k'=1}^{k_n}\sum_{\substack{k''=1\\k''>k'}}^{k_n}\frac{p_{nk'}^{+}+p_{nk'}^{-}}{1-q_{nk'}^{0}}\,\frac{p_{nk''}^{+}+p_{nk''}^{-}}{1-q_{nk''}}\,\frac{1-q_{nk'}^{0}}{q_{nk'}^{+}+q_{nk'}^{-}}\,\frac{1-q_{nk''}^{0}}{q_{nk''}^{+}+q_{nk''}^{-}}\right) = 0$$

or

$$\sum_{k=1}^{k_n}\left(\frac{p_{nk}^{+}+p_{nk}^{-}}{q_{nk}^{+}+q_{nk}^{-}}\right)^2 = 0.$$

Hence the condition (1) is satisfied.

Let remember connections (7) and (8), then we get:

$$\lim_{n\to\infty}\frac{P(\sum_{k=1}^{k_n}\eta_{nk}=1)}{P(\sum_{k=1}^{k_n}\eta_{nk}=0)} = \lambda+\mu \quad \text{and} \quad \lim_{n\to\infty}\sum_{k=1}^{k_n}\frac{p_{nk}^{+}+p_{nk}^{-}}{q_{nk}^{+}+q_{nk}^{-}} = \lambda+\mu.$$

Since

$$\sum_{k=1}^{k_n} \frac{p_{nk}^+ + p_{nk}^-}{1 - q_{nk}^0} - \sum_{k=1}^{k_n} \frac{p_{nk}^+ + p_{nk}^-}{q_{nk}^+ + q_{nk}^-} \leqslant \max_{1 \leqslant k \leqslant k_n} \frac{p_{nk}^+ + p_{nk}^-}{1 - q_{nk}^0} \sum_{k=1}^{k_n} \frac{p_{nk}^+ + p_{nk}^-}{q_{nk}^+ + q_{nk}^-},$$

and condition (1) is proved, we get condition (4). It remains to prove condition (5). Because

$$\omega_n^1(t) = \prod_{k=1}^{k_n} \left((q_{nk}^+ + p_{nk}^+) - (q_{nk}^- + p_{nk}^-) \right)$$
$$\times \exp\left\{ (e^{it} - 1) \left(\sum_{k=1}^{k_n} \frac{p_{nk}^+ + p_{nk}^-}{1 - q_{nk}^0} - 2 \sum_{k=1}^{k_n} \frac{q_{nk}^+ p_{nk}^- - q_{nk}^- p_{nk}^+}{(q_{nk}^+ + p_{nk}^+)^2 - (q_{nk}^- + p_{nk}^-)^2} \right) \right.$$
$$\left. + B \max_{1 \leqslant k \leqslant k_n} \frac{p_{nk}^+ + p_{nk}^-}{1 - q_{nk}^0} \sum_{k=1}^{k_n} \frac{p_{nk}^+ + p_{nk}^-}{1 - q_{nk}^0} \right\} \to \alpha_1 e^{(\lambda - \mu)(e^{it} - 1)},$$

and using above proved limits (3) and (4), we get

$$\alpha_1 \left(\lim_{n \to \infty} \sum_{k=1}^{k_n} \frac{q_{nk}^+ p_{nk}^- - q_{nk}^- p_{nk}^+}{(q_{nk}^+ + p_{nk}^+)^2 - (q_{nk}^- + p_{nk}^-)^2} - \mu \right) = 0,$$

and condition (5) is satisfied. Theorem is proved.

References

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Nepriklausomų atsitiktinių dydžių sandaugų konvergavimas į logpuasono dėsnį

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Darbe nagrinėjamas serijų sekos nepriklausomų kiekvienoje serijoje atsitiktinių dydžių sandaugų M-silpnas konvergavimas į logpuasono dėsnį. Yra įrodoma teorema su penkiom sąlygom.