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Asymptotic expansion for the distribution function of the series scheme of random variables in the large deviation Cramer zone

Dovilė DELTUVIENĖ (VGTU)
e-mail: lsaulis@fm.vtu.lt

Among the problems of limit theorems, the theorems for large deviations play an important part in the case where separate summands of the sums of random variables satisfy the Cramer or Linnik conditions. Most of principal ideas and rezults in this field are rather exhaustively described in the monographs [1]–[3]. Asymptotic expansion of sums of identically distributed random variables have been investigated by A. Bikelis, A. Žemaitis, Š. Jakševičius.

This work is designated for obtaining asymptotic expansions and determination of structures of the remainder terms that take into consideration large deviations both in the Cramer zone for the distribution function of sums of independent random variables in the series scheme. The result was obtained, based on L. Saulis General Lemma 1 [4] by joining the methods of characteristic functions and cumulants. This work extends Book's [6] results to the sums of random variables with weights.

Let $\xi_j^{(n)}$, $j = 1, 2, \ldots, n$, be a triangular array of independent random variables (r.v.) with means $\mathbf{E}\xi_j^{(n)} = 0$, and variances ${\sigma_j^{(n)}}^2 = \mathbf{E}{\xi_j^{(n)}}^2$. Put

$$S_n = \sum_{j=1}^{n} \xi_j^{(n)}, \quad Z_n = B_n^{-1} S_n, \quad B_n^2 = \sum_{j=1}^{n} {\sigma_j^{(n)}}^2, \tag{1}$$

$$F_n(x) = \mathbf{P}(Z_n < x), \quad \psi(x) = \frac{\varphi(x)}{1 - \Phi(x)}, \tag{2}$$

where $\Phi(x)$ and $\varphi(x)$ is $N(0,1)$ normal distribution and its density, respectively. Next, let $f_\xi(t)$ and $\Gamma_k(\xi)$ denote the characteristic function (ch.f.) and kth-order cumulant of r.v. ξ.

We have to obtain estimate of kth-order cumulants $\Gamma_k(Z_n)$, $k = 3, 4, \ldots$, of r.v. Z_n in order that we could use L. Saulis General Lemma 1 [4] for large deviations.

Let us say that r.v's $\xi_j^{(n)}$, $j = 1, 2, \ldots, n$, are subject to condition (B), if there exists quantity $K_j^{(n)} > 0$ such that

$$|\mathbf{E}(\xi_j^{(n)})^k| \leqslant k! (K_j^{(n)})^{k-2} {\sigma_j^{(n)}}^2, \quad k = 3, 4, \ldots . \tag{B}$$

If condition (B) is fulfilled, then, for the kth-order cumulants $\Gamma_k(\xi_j^{(n)})$ of the r.v. $\xi_j^{(n)}$, the following estimate

$$|\Gamma_k(\xi_j^{(n)})| \leqslant k!\left(2\big(K_j^{(n)} \vee \sigma_j^{(n)}\big)\right)^{k-2}, \quad k = 3, 4, \ldots \tag{3}$$

holds. Here and in what follows $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, $[m]$ is an integer part of number m. The proof of inequality (3) is given in [3] on p. 42.

PROPOSITION 1. If for r.v. $\xi_j^{(n)}, j = \overline{1, n}$, condition (B) is fulfilled, then for the kth-order cumulant $\Gamma_k(Z_n)$ of the r.v. Z_n, the estimate

$$|\Gamma_k(Z_n)| \leqslant k!\Delta_n^{2-k}, \quad k = 3, 4, \ldots \tag{4}$$

holds, where

$$\Delta_n = K_n^{-1}B_n, \quad K_n := 2\max_{1\leqslant j\leqslant n}\left(K_j^{(n)} \vee \sigma_j^{(n)}\right). \tag{5}$$

We further suppose that $\Delta_n \to \infty$, as $n \to \infty$.

Proof is analogous to the proof of Proposition in [7].
Denote:

$$\Delta_{n,0} := c_0\Delta_n^{1/(1+2\gamma)}, \quad c_0 = (1/6)(\sqrt{2}/6), \quad R_n = (1/12)(1 - x/\Delta_n)\Delta_n. \tag{6}$$

Further, suppose that the densities $p_{\xi_j^{(n)}}(x)$ (if exists) are bounded, i.e.,

$$\sup_x p_{\xi_j^{(n)}}(x) = C_j^{(n)} < \infty. \tag{D}$$

In case $\xi_j^{(n)}$ has no density, then $C_j^{(n)} = \infty$ by definition.

Theorem 1. *If for the r.v. $\xi_j^{(n)}$, $j = \overline{1, n}$, with $\mathbf{E}\xi_j^{(n)} = 0$ and ${\sigma_j^{(n)}}^2 > 0$, $j = \overline{1, n}$, conditions* (B) *and* (D) *are fulfilled, then for each integer l, $l \geqslant 3$, in the Cramer zone $0 \leqslant x < \Delta_{n,0}$, the asymptotic expansion*

$$\begin{aligned}\frac{1 - F_{Z_n}(x)}{1 - \Phi(x)} &= \exp\{L_n(u)\}\left\{\frac{\psi(x)}{\psi(u)}\left(1 + \sum_{\nu=1}^{l-3} L_{\nu,n}(u)\right) + \theta_1 8\sqrt{2\pi}(x+1)\right.\\ &\times\left[\frac{c(l,x)}{\Delta_n^{l-2}} + \frac{285\Delta_n \exp\{-(1 - x/\Delta_{n,0})\sqrt{\Delta_{n,0}}\}}{(1 - x/\Delta_{n,0})} + \theta_2\left(\frac{\pi^2}{2T_{n,0}^2}\exp\left\{-\frac{1}{\pi^2}T_{n,0}^2\right\}\right.\right.\\ &\left.\left.\left.+\frac{CK_n}{\Delta_n}\max_{1\leqslant r_i\leqslant n}\prod_{i=1}^{4} {C_{r_i}^{(n)}}^{1/4}\exp\left\{-\frac{c_3}{K_n^2}\sum_{k=1}^{n} {C_k^{(n)}}^{-2}\right\}\right)\right]\right\}\end{aligned} \tag{7}$$

holds. Here $L_n(x) = \sum_{k=3}^{\infty} \lambda_{k,n} x^k$, *where the coefficients* $\lambda_{k,n}$ *are found by formula* (5) [4]. *For the coefficients* $\lambda_{k,n}$ *the estimate* $|\lambda_{k,n}| \leqslant (2/k)(16/\Delta_n)^{k-2}$, $k = 3, 4, \ldots$ *holds. The function* $\psi(x)$ *defined by equality* (2). *The quantity*

$$u_n(x) = x\left(1 + \sum_{k=1}^{l-3} c_{k,n} x^k + \theta c^*(l)(x/\Delta_n)^{l-2}\right), \tag{8}$$

where $c^*(l) = 736l(l-1)(7/2)^{l-2}l!$ *and the coefficients* $c_{k,n}$ *are expressed by the cumulants of the r.v.* Z_n *and found by formula* (11) [4].

Polynomials $L_{\nu,n}(u)$ *are determined by relation* (104) [4]. *In particular,*

$$\begin{aligned} L_{1,n}(u_n(x)) &= -\frac{1}{2}\Gamma_3(Z_n)\frac{1}{x} + \frac{3}{2}\left(2\Gamma_4(Z_n) - 3\Gamma_3^2(Z_n)\right) \\ &\quad + \frac{1}{48}\left(72\Gamma_5(Z_n) - 394\Gamma_3(Z_n)\Gamma_4(Z_n) + 267\Gamma_3^3(Z_n)\right)x + \ldots, \\ L_{2,n}(u_n(x)) &= \frac{1}{24}\left(3\Gamma_4(Z_n) - 5\Gamma_3^2(Z_n)\right) \\ &\quad + \frac{1}{24}\left(3\Gamma_5(Z_n) - 16\Gamma_3(Z_n)\Gamma_4(Z_n) + 15\Gamma_3^3(Z_n)\right)x + \ldots. \end{aligned}$$

The fool expressions of the quantities $c(l, x) \sim C(l)x^{l-3}$ *is extracted by* (9) [4].

Proof. Let $\xi_j^{(n)}(h)$, $j = 1, 2, \ldots, n$, be r.v. conjugate to the $\xi_j^{(n)}$ with the distribution density

$$p_{\xi_j^{(n)}(h)}(x) = \mathrm{e}^{hy} p_{\xi_j^{(n)}}(y) \Big/ \int_{-\infty}^{\infty} \mathrm{e}^{hy} p_{\xi_j^{(n)}}(y)\mathrm{d}y. \tag{9}$$

Then

$$m_j(h) = \mathbf{E}\xi_j^{(n)}(h) = \sum_{k=2}^{\infty} \frac{1}{(k-1)!}\Gamma_k(\xi_j^{(n)})h^{k-1}, \tag{10}$$

$$\sigma_j^{(n)^2}(h) = \mathbf{D}\xi_j^{(n)}(h) = \sum_{k=2}^{\infty} \frac{1}{(k-2)!}\Gamma_k(\xi_j^{(n)})h^{k-2}, \tag{11}$$

and the kth-order cumulant of the r.v. $\xi_j^{(n)}(h)$ is

$$\Gamma_k\left(\xi_j^{(n)}(h)\right) = \sum_{l=k}^{\infty} \frac{1}{(l-k)!}\Gamma_l(\xi_j^{(n)})h^{l-k}. \tag{12}$$

Put

$$S_n(h) = \sum_{j=1}^{n} \xi_j^{(n)}(h), \quad Z_n(h) = \frac{S_n(h) - M_n(h)}{B_n(h)},$$

$$L_{k,n}(h) = B_n^{-k}(h) \sum_{j=1}^{n} \mathbf{E} \left| \xi_j^{(n)}(h) - m_j(h) \right|^k, \tag{13}$$

where $M_n(h) = \mathbf{E} S_n(h) = \sum_{j=1}^n \mathbf{E}\xi_j^{(n)}(h)$, $B_n^2(h) = \sum_{j=1}^n {\sigma_j^{(n)}}^2(h)$ and the quantity h is determined from the equation $x = M_n(h)/B_n$. If we put the conjugate r.v. $\xi_j^{(n)}(h)$ instead of $\xi_j^{(n)}$ in relation $l_n(N_n) = B_n^{-2} \sum_{j=1}^n \int_{|x| \leqslant N_n} x^2 p_{\tilde{\xi}_j(x)} \mathrm{d}x$, $N_n > 0$ then

$$l_n\left(N_n(h)\right) \geqslant 2\left(1 - 2B_n^2(h) L_{4,n}(h)/N_n^2(h)\right), \tag{14}$$

here $p_{\tilde{\xi}_j}(x)$ is the density of a symmetrized r.v. $\tilde{\xi} = \xi - \xi'$, where ξ' is a r.v. independent of ξ with the same distribution as the r.v. ξ. Using the relation $\mathbf{E}\left(\xi_j^{(n)}(h) - m_j(h)\right)^4 = \Gamma_4\left(\xi_j^{(n)}(h)\right) + 3{\sigma_j^{(n)}}^4(h)$, we find

$$L_{4,h}(h) \leqslant \Gamma_4\left(S_n(h)\right)/B_n^4(h) + 3 \max_{1 \leqslant j \leqslant n} {\sigma_j^{(n)}}^2(h)/B_n^2(h). \tag{15}$$

By virtue of inequality (3) and relation (15), when $0 \leqslant h \leqslant \Delta_n/(12B_n)$ we get

$${\sigma_j^{(n)}}^2(h) = {\sigma_j^{(n)}}^2\left(1 + \theta \sum_{k=3}^{\infty} k(k-1)\left(\frac{hB_n}{\Delta_n}\right)^{k-2}\right) = {\sigma_j^{(n)}}^2\left(1 + \theta\frac{5}{8}\right). \tag{16}$$

Further, according to (16), we have

$$\begin{aligned} \Gamma_4\left(S_n(h)\right) &= \sum_{k=4}^{\infty} \frac{1}{(k-4)!} \Gamma_k(S_n) h^{k-4} \\ &\leqslant (3.5K_nB_n)^2 \sum_{k=4}^{\infty} (k-2)(k-3)(3.5hK_n)^{k-4} \leqslant 81(3.5K_nB_n)^2. \end{aligned}$$

Hence, $\Gamma_4\left(S_n(h)\right)/B_n^4(h) \leqslant 18.2(B_n/\Delta_n)^2/B_n^2(h)$. Employing (15) we obtain

$$L_{4,n}(h) B_n^2(h) \leqslant 18.5(B_n/\Delta_n)^2. \tag{17}$$

Put $N_n(h) = 6.2(B_n/\Delta_n)$, then (14) implies $l_n\left(N_n(h)\right) \geqslant 1$. Then employing Lemma 6.5 [3], we obtain

$$\left|f_{Z_n(h)}(t)\right| \leqslant \exp\left\{-\frac{1}{\pi^2}t^2\right\}, \quad |t| \leqslant T_{n,0}, \quad T_{n,0} = (\pi/15)(\Delta_n/B_n). \tag{18}$$

By virtue of L. Saulis General Lemma 1[4], we need to estimate the integral

$$I = \int_{R_n}^{T_n} \left|f_{Z_n(h)}(t)\right| \frac{\mathrm{d}t}{t}, \tag{19}$$

where $T_n = C(l)\Delta_n^{l-2}$, Δ_n and R_n are defined by equalities (6). Put $I = I_1 + I_2$,

$$I_1 = \int_{R_n}^{T_{n,0}} \left|f_{Z_n(h)}(t)\right| \frac{dt}{t}, \quad I_2 = \int_{T_{n,0}}^{T_n} \left|f_{Z_n(h)}(t)\right| \frac{dt}{t}.$$

From inequality (18) we get

$$I_1 = \int_{R_n}^{T_{n,0}} \left|f_{Z_n(h)}(t)\right| \frac{dt}{t} \leqslant \left(\pi^2/2R_n\right) \exp\left\{ - R_n^2/\pi^2\right\}. \tag{20}$$

Now, employing Lemma 6.6 [3], we find estimate of the integral

$$\begin{aligned} I_2 &= \int_{T_{n,0}}^{T_n} \left|f_{Z_n(h)}(t)\right| \frac{dt}{t} = 2\pi B_n(h) \int_{(2N_n(h))^{-1} \leqslant |t| < T_n} \left|f_{S_n(h)}(2\pi t)\right| \frac{dt}{t} \\ &\leqslant 2\pi e^4 \sqrt{2\pi} \exp\left\{ - \frac{c_3}{K_n^2} \sum_{j=1}^{n} C_j^{(n)^{-2}} \right\} U_n(h), \end{aligned} \tag{21}$$

where $c_3 > 0$, and $U_n(h)$ by Cauchy's inequality, is

$$\begin{aligned} U_n(h) &= \sum_k \sup_{t_k^{(n)}} < t < t_{k+1}^{(n)} \left|f_{S_n(h)}(2\pi t)\right| \\ &\leqslant \prod_{i=1}^{4} \left(\sum_k \sup_{t_k^{(n)} < t < t_{k+1}^{(n)}} \left|f_{\xi_j^{(n)}(h)}(2\pi t)\right|^4 \right)^{1/4} \leqslant 172 K_n \prod_{i=1}^{4} C_i^{(n)^{1/4}}. \end{aligned} \tag{22}$$

From inequalities (21) and (22), we get

$$I_2 = \int_{T_{n,0}}^{T_n} \left|f_{Z_n(h)}(t)\right| \frac{dt}{t} \leqslant 684 e^4 \pi \sqrt{2\pi} K_n \max_{1 \leqslant r_i \leqslant n} \prod_{i=1}^{4} C_{r_i}^{(n)^{1/4}} \exp\left\{ - \frac{c_3}{K_n^2} \sum_{j=1}^{n} C_j^{(n)^{-2}} \right\}.$$

Finally, making use this estimates and inequality (20) we get the proof of Theorem 1.

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Atsitiktinių dydžių sumos serijų schemoje pasiskirstymo funkcijos asimptotinis skleidinys didžiųjų nuokrypių Kramerio zonoje

D. Deltuvienė

Darbas skirtas nepriklausomų atsitiktinių dydžių (at.d.) $\xi_j^{(n)}$, $j = \overline{1,n}$, su vidurkiais $\mathbf{E}\xi_j^{(n)} = 0$ ir dispersijomis $\sigma_j^{(n)2} = \mathbf{E}\xi_j^{(n)2}$ serijų schemoje sumos pasiskirstymo funkcijos asimptotinio skleidinio gavimui didžiųjų nuokrypių Kramerio zonoje. Rezultatas gautas remiantis L. Saulio bendrąja Lema 1 [4], apjungiant charakteristinių funkcijų ir kumuliantų metodus. Darbas atskiru atveju praplečia at.d. sumavimo su svoriais rezultatus, gautus S.A. Book [6].