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# Decidability of a monadic subclass of modal logic S4

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## 1. Introduction

We will consider the formulas of quantifier modal logic S4. We will denote the formulas by $F, G, H, F', G', H', F_1, G_1, H_1, \ldots$, literals of classical logic by $L, L_1, L_2, \ldots$, predicate variables by $P, R, P_1, R_1, \ldots$, propositional variables by $p, q, r, p_1, q_1, r_1, \ldots$, and finite or empty list of formulas of the language considered by $\Gamma, \Delta, \Gamma_1, \Delta_1, \ldots$. The order in $\Gamma$ will always be disregarded, hence $\Gamma$ is treated as a multiset.

G. Mints described in [1] a reduction of an arbitrary formula $F$ to a finite set of such formulas $G_1, G_2, \ldots, G_s$ that $\vdash F$ is derivable in S4 if and only if $G_1, G_2, \ldots, G_s \vdash$ is derivable in S4. Moreover, the formula $G_i (i = 1, 2, \ldots, s)$ have one of the following forms

$$\Box Q_1 x_1 \ldots Q_n x_n (L_1 \vee \Box L_2), \Box Q_1 x_1 \ldots Q_n x_n (L_1 \vee \Diamond L_2), \Box Q_1 x_1 \ldots Q_n x_n (L_1 \vee L_2),$$
$$\Box Q_1 x_1 \ldots Q_n x_n (L_1 \vee L_2 \vee L_3), L, \tag{1}$$

where $Q_i \in \{\forall, \exists\}$.

In this paper using the reduction of G. Mints we will describe one decidable monadic subclass of modal logic S4 and two undecidable classes. Some decidable monadic subclasses presented in [2], [3], [4].

## 2. Decidability

It is well known that the monadic class of the modal logic S4 is undecidable. We will prove decidability of a class of closed formulas, that is, the formulas do not contain free individual variables.

DEFINITION 1. We denote by $E$ a class of closed formulas containing only one-place predicate variables. Moreover, the formulas of the class the following conditions hold:

1) the formulas $F$ contain only logical connectives $\neg, \wedge, \vee$ and no logical or modal symbol in $F$ occur in the scope of a negation,

2) each subformula of the form $\Box G(\Diamond G)$, that is, a subformula beginning with modal operator does not contain any free individual variable or, otherwise, a subformula $G$ is quantifier-free and contains only one free individual variable.

For example, the following formulas belong to the class $E$:

$$\Box\forall y P(y) \wedge \Diamond\forall x\exists y\forall z((P(x) \wedge \neg R(y)) \vee (P(y) \wedge R(z)))$$
$$\Diamond\forall x\forall y(\Box(P(x) \vee \neg R(x)) \wedge \Diamond(R(y) \vee \neg P(y))).$$

The next formula does not belong to the class $E$:

$$\Box\forall x\Box\forall y(P(x) \vee Q(y)).$$

DEFINITION 2. If $F$ is a subformula of a modal formula $G$, then the modal degree of $F$ in $G$ is equal to the number of modal operators governing $F$.

DEFINITION 3. Modal literals are the expressions of the form $L, \Box L, \Diamond L$. Modal clauses are disjunctions of modal literals.

DEFINITION 4. (see [2]). By Near–Monadic we denote the class of formulas without function symbols such that no occurrence of a subformula contains more than one free individual variable.

DEFINITION 5. A formula $F$ will be called a formula with small clauses if each subformula of $F$ of the form $G_1 \vee G_2 \vee ... \vee G_s$ contains at most three terms.

In that follow, $\forall(\exists)D_i$ denotes a formula in prenex normal form containing only one occurrence of the quantifiers $\forall, \exists$.

**Theorem 1.** *Class E is decidable.*

*Proof.* First of all, we will show that for any formula $F$ of the class $E$ one can find the closed formulas with short clauses $\forall(\exists)D_1, ..., \forall(\exists)D_s$ and a propositional variable for which the following condition holds: a sequent $\vdash F$ is derivable in S4 if and only if $\Box\forall(\exists)D_1, ..., \Box\forall(\exists)D_s, \neg p \vdash$ is derivable in S4. Moreover, the formulas $D_1, ..., D_s$ contain only one-place predicate and propositional variables. Suppose that a modal degree of a subformula $\Box G$(or $\Diamond G$) of a formula $F$ is equal to 1. In this case a formula $G$ can be:

1) a quantifier-free formula containing only one free variable (denote it by $x$),
2) a closed formula.

In the first case we change the formula $\Box G(\Diamond G)$ by introduction of new predicate variable $P(x)$ not occurring in a formula $F$. In the second case we transform at first a formula $G$ and denote the obtained formula by $G'$.

We transform a formula $G$ into $G'$ by applying the classical equivalences

$$\forall x(F \wedge G) \equiv \forall xF \wedge \forall xG, \forall x(F \vee H) \equiv \forall xF \vee H, \exists x(F \wedge H) \equiv \exists F \wedge H,$$
$$\exists x(F \vee G) \equiv \exists xF \vee \exists xG. \tag{2}$$

The variable $x$ does not occur in a formula $H$. One needs to use also the propositional classical equivalences for reducing the subformulas into disjunctive and conjunctive normal forms.

Obtained formula $G'$ the following conditions hold:

1) each occurrence of $\forall x$ (where $x$ is an individual variable) is only immediate before a formula of the form $L_1(x) \vee ... \vee L_j(x)$,
2) each occurrence of $\exists x$ is only immediate before a formula of the form $L_1(x) \wedge ... \wedge L_j(x)$.

We change the subformula $\Box G'(\Diamond G')$ in a formula $F$ by introduction of new predicate variable and we denote the obtained formula by $F'$. Now we have the following cases:

1) the formula $F'$ does not contain the modal operators,
2) the formula $F'$ has less modal operators than the old one.

In the second case there exists an occurrence in $F'$ of a subformula of the form $\Box G(\Diamond G)$ (where $G$ does not contain the modal operators). In this case we repeat the same transformation described so far. Therefore we may successively eliminate all modal operators. After a transformation we change each new predicate variable by an initial corresponding formula. We use $H$ for obtained formula.

Since $H$ is obtained by applying only the classical propositional equivalences and equivalences (2), the formulas $F$ and $H$ are deductive equivalents. In fact, assume that a formula $G'$ is obtained from an arbitrary formula $G$ of modal logic S4 by applying a classical propositional equivalence or an equivalence of (2). We transform the formulas $G, G'$ into the formulas of classical predicate logic using the method presented by A. Nonnengart in [5]. Both obtained formulas of classical predicate logic with two sorts of individual variables are equivalents. This implies that the formulas $G, G'$ are deductive equivalents.

We will transform the formula $H$ into a sequence of closed formulas $\Box\forall(\exists)D_1, ..., \Box\forall(\exists)D_s, \neg p$ (where $D_i$ is a modal clause) such that a sequent $\vdash H$ is derivable in S4 if and only if $\Box\forall(\exists)D_1, ..., \Box\forall(\exists)D_s, \neg p \vdash$ is derivable in S4. Moreover, the formulas $D_1, ..., D_s$ contain only one-place predicate and propositional variables. We will use a method presented in G. Mints [1]. The formula $H$ contains a negation only immediate before atomic formulas. From this follows that a reduction of formula $H$ to the sequence of formulas use the monotonic positive replacement instead of equivalent replacement (see [1], [2]). In what follow, *A,B,D* denote one-place predicate variables.

1) If a subformula $G$ is of the form $B(x) \vee D(x)$, then we replace $G$ by a new predicate variable $A(x)$ and we add the formulas $\Box\forall x(A(x) \vee \neg B(x))$, $\Box\forall x(A(x) \vee \neg D(x))$ in an antecedent of the considered sequent.

2) If $G \equiv B(x) \wedge D(x)$, then we add the formula $\Box\forall x(A(x) \vee \neg B(x) \vee \neg D(x))$.
3) If $G \equiv \neg B(x)$, then we add the formula $\Box\forall x(A(x) \vee B(x))$.
4) If $G \equiv \forall x B(x)$, then we add the formula $\Box\exists x(q \vee \neg B(x))$, where $q$ is a new propositional variable.
5) If $G \equiv \exists x B(x)$, then we add the formula $\Box\forall x(q \vee \neg B(x))$, where $q$ is a new propositional variable.
6) If $G \equiv \Box B(x)$, then we add the formula $\Box\forall x(A(x) \vee \Diamond\neg B(x))$.
7) IF $G \equiv \Diamond B(x)$, then we add the formula $\Box\forall x(A(x) \vee \Box\neg B(x))$.

We introduce similarly a replacement in the case when a subformula $G$ contains the propositional variables. The obtained formula $\Box\forall(\exists)D_1 \wedge ... \wedge \Box\forall(\exists)D_s \wedge \neg p$ belongs to a near-monadic class. This class is decidable (see T. Tammet [2]). Theorem is proved.

## 3. Undecidability

DEFINITION 6. Given a formula $F$, the formula obtained from $F$ by deleting all occurrences of modal logic operators, is called a projection of $F$.

DEFINITION 7. Given a set $M$ of formulas, the set of projections of all formulas of $M$ is called a projection of the set $M$.

A set of the sequents $G_1, G_2, ..., G_s \vdash$, where $G_i (i = 1, ..., s)$ have one of the forms (1) is undecidable. In other words a class of the formulas of the form $G_1 \wedge ... \wedge G_s$, where $G_i (i = 1, ..., s)$ have one of the forms (1) is undecidable. Modal clauses in (1) contain at most three terms, that is, the formulas of the form (1) are the formulas with small clauses.

**Theorem 2.** *A class of formulas of the form $G_1 \wedge ... \wedge G_s$, where $G_i (i = 1, ..., s)$ are the formulas of the form (1) and only one of them contains a modal clause with three terms is undecidable.*

*Proof.* Clearly, if any class $X$ of formulas of classical predicate logic is undecidable, then the class of modal logic formulas $F$ such that $pr(F) \in X$ is also undecidable. Without loss of generality we consider the formulas containing a negation only immediate before atomic formulas. From this follows that a reduction, presented by G. Mints, of any formula to the sequence of formulas use only monotonic positive replacements. The list of obtained formulas of the form $\Box Q_1 x_1 ... Q_n x_n D$ $(Q_i \in \{\forall, \exists\})$ contain the clauses of three terms only in the case, then there exists a replacement of a conjunction of two subformulas by introduction of new variable (see the proof of Theorem 1).

Hence, if there exists an undecidable class by derivability in classical predicate logic whose formulas contain at most one occurrence of conjunction and a negation is only immediate before atomic formulas, then the Theorem is valid. Such class is presented by V.P. Orevkov in [6]. The class of formulas of classical logic of the form

$$\forall x_1 ... \forall x_2 \exists y \forall z \exists u_1 \exists u_2 \exists u_3 \exists u_4 (D_1 \wedge D_2),$$

where $D_1, D_2$ are the clauses (the clauses contain a disjunction and a negation is only immediate before atomic formulas), is a reduction class by derivability. Theorem is proved.

**Theorem 3.** *There exists a class of closed formulas without function symbols and propositional variables of the form*

$$\exists x_1 ... \exists x_n (\Box \forall y_1 ... \forall y_m \exists z_1 ... \exists z_k (D_1 \wedge ... \wedge D_s) \wedge L),$$

*(where $D_i$ $(i = 1, ..., s)$ is a modal clause, L is a literal of classical logic), which is undecidable in the modal logic S4, but its projection is decidable in clasical predicate logic.*

*Proof.* In [4] is proved that a class of closed formulas without function symbols of the form

$$\Box \forall y_1 ... \forall y_m \exists z_1 ... \exists z_k (D_1 \wedge ... \wedge D_s) \wedge l \quad (3)$$

(where $D_i$ is a modal clause which can contain also propositional variables, $l$ is a propositional literal) is undecidable in modal logic S4 and its projection is decidable in classical predicate logic.

Suppose that a formula of the form (3) contains $m$ different propositional variables $p_1, ..., p_m$. We change $p_i$ $(i = 1, ..., m)$ by new one-place predicate variable $P_i(x_i)$ ($x_i$ is a new individual variable) and we add an expression $\exists x_1 ... \exists x_m$ in front of the considered formula. The obtained formula is deductive equivalent to an initial formula. Theorem is proved.

## References

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## Vienos modalumo logikos S4 klasės su vienviečiais predikatiniais kintamaisiais išsprendžiamumas

S. Norgėla

Darbe naudojamasi žinoma G. Mints kvantorinės modalumo logikos S4 formulių transformacija. Įrodomas vienos klasės su vienviečiais predikatiniais kintamaisiais išsprendžiamumas. Formulėse esantys disjunktai turi ne daugiau kaip tris narius.