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Realization of operator expressions for the solutions of differential equations

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1. Introduction

It should be emphasized that the operator calculus algorithms become more and more popular this time. Operator solution of nonlinear differential equations and of their systems is applied together with other popular methods. That is why one needs to create new specialized algorithms and to test their efficiency.

In this work we present the computer realization of operator algorithms and some examples. The results obtained are applied to solving problems of vibration analysis.

2. The general part

We are going to use linear operators specified by:

$$\mathbf{D}_x x^n \overset{def}{=} n x^{n-1}, \quad n=0,1,2,\ldots; \quad \text{then } \mathbf{D}_x f=(f)'_x \text{ and } \mathbf{1}f \overset{def}{=} f,\ \mathbf{D}_x^0 \overset{def}{=} \mathbf{1}.$$

Let a differential equation $y''_{xx} = P(x, y, y'_x)$ with initial conditions $y(x; s, t, v)|_{x=v} = s$; $(y(x; s, t, v))'_x|_{x=v} = t$ be given. The expression of the solution is of the form:

$$y = y(x; s, t, v) = \sum_{k=0}^{+\infty} p_k(s, t, v)\frac{(x-v)^k}{k!}, \quad v \in R; \tag{1}$$

here $p_k(s,t,v) = (\mathbf{D}_v + t\mathbf{D}_s + P(v,s,t)\mathbf{D}_t)^k s$, where $P(x,s,t)$ is either a polynomial or a function of variables x, s and t , such that the obtained series in a chosen interval converge absolutely (for instance, the coefficients p_k satisfy the condition $|p_k(s,t,v)| \leqslant M^k$), [1].

The above solution (expression (1)) can be also presented in the form

$$y(x; s, t, v) = \sum_{k=0}^{+\infty} p_k(s_l, t_l, v)|_{v=v_l}\frac{(x-v_l)^k}{k!} \overset{def}{=} y_l(x; s, t, v), \quad l = 1, 2, \ldots, \tag{2}$$

provided

$$s_{l+1} = \sum_{k=0}^{+\infty} p_k(s_l, t_l, v_l)\frac{(v_{l+1}-v_l)^k}{k!},$$
$$t_{l+1} = \sum_{k=0}^{+\infty} p_{k+1}(s_l, t_l, v_l)\frac{(v_{l+1}-v_l)^k}{k!}. \tag{3}$$

The sequence $v_2, v_3, \ldots$ is chosen arbitrarily, but $v_1 = v$, $s_1 = s$, $t_1 = t$.

It is possible to generalize this approach by including differential equations of other orders as well as their systems.

3. The computing strategy

The solution (1) is a series in x with coefficients p_k , which, in their turn, are functions of the initial conditions and the center v. Any computer realization gives only a finite number of coefficients $p_0, p_1, \ldots, p_N$ and, every time, we get one or another approximation of the solution.

Having functions $p_k(s,t,v)$, $k = 0, \ldots, N$, we construct a polynomial $\hat{y}(x;s;t,v) = \sum_{k=0}^{N} p_k(s,t,v)\frac{(x-v)^k}{k!}$. Substituting concrete numerical values for the variables s, t and v, we get an approximation of the solution – the polynomial $\hat{y}_1(x)$ in the neighborhood of a point v . Using expressions (2) and (3) and summing up to N, we get a family of approximations, i.e., the polynomials $\hat{y}_l(x)$ with centers v_l. The latter approximations "go away" from the exact solution, as the variable x "moves away" from the center. In connection with this, the approximation $y^*(x)$ (Fig. 2) of the solution is to be formed, using the family of approximations $\hat{y}_l(x)$ (Fig. 1), as follows:

$$y^*(x) = \hat{y}_l(x), \quad \text{for} \quad v_l \leqslant x < v_{l+1},\ l = 1, 2, \ldots, n.$$

The same methodics is used for solving of ordinary differential equations and their systems.

Computer realization is done using software *Maple*. In order to calculate the functions p_k, the symbolic differentiation is used – the function *diff*. Approximation of the solution and analogous approximations are formed using the function *piecewise* and the

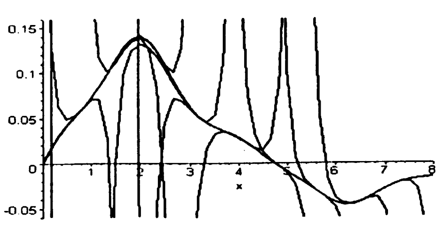

Fig. 1. The family of approximations – polynomials.

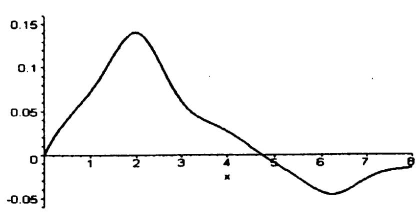

Fig. 2. The approximation of a solution.

function *op*. All drawings are made using the function *plot*. The symbolic simplification – the function *simplify*, the summation *sum* and the cycle *for* are also in use. It is possible to choose the step size of center variation, the degree N of the polynomial, the numerical values of the parameters and the number n of polynomials, used in forming the solution approximations. If one make use of the function *digits*, the accuracy of calculations increases, [3].

4. Analysis of the method

The preliminary analysis of the method is presented below.

Direct symbolic analysis

Let a differential equation $y' = y^2$, $y(v) = s$, be given. Its analytical solution looks like $y(x,s,v) = \frac{s}{1-s(x-v)}$, and the operator expression of the solution is

$$y = y(x;s,v) = \sum_{k=0}^{\infty} p_k \frac{(x-v)^k}{k!}, \quad \text{provided } p_k = p_k(s,v) = (\mathbf{D}_v + s^2\mathbf{D}_s)^k s.$$

Then $p_0 = s$, $p_1 = 1 \cdot s^2$, $p_2 = 1 \cdot 2s^3, \ldots, p_n = n! s^{n+1}$, i.e.,

$$y = y(x;s,v) = \sum_{k=0}^{\infty} k! s^{k+1} \frac{(x-v)^k}{k!} = s \sum_{k=0}^{\infty} (s(x-v))^k.$$

Thus,

$$y'_x = \sum_{k=0}^{+\infty} p_{k+1} \frac{(x-v)^k}{k!} = \sum_{k=0}^{\infty} (k+1)! s^{k+2} \frac{(x-v)^k}{k!} = s^2 \sum_{k=0}^{\infty} (k+1)(s(x-v))^k;$$

$$y^2 = s^2 \left(1 + s(x-v) + (s(x-v))^2 + \ldots\right)^2$$
$$= s^2 \left(1 + 2s(x-v) + 3s^2(x-v)^2 + \ldots\right) = s^2 \sum_{k=0}^{+\infty} (k+1)(s(x-v))^k.$$

So, we have $y' - y^2 = 0$.

Taking a finite sum, we get $y^*(x,s,v) = s \sum_{k=0}^{N} (s(x-v))^k$. Now,

$$y'^*_x = s^2 \sum_{k=1}^{N} k(s(x-v))^{k-1} = s^2 \sum_{k=0}^{N-1} (k+1)(s(x-v))^k,$$

$$y^{*2} = s^2 \left(\sum_{k=0}^{N} (s(x-v))^k \right)^2$$
$$= s^2 \left(1 + s(x-v) + s^2(x-v)^2 + \ldots + s^N(x-v)^N\right)^2$$

$$= s^2\big(1 + 2s(x-v) + 3s^2(x-v)^2 + \ldots + (N+1)s^N(x-v)^N\big)$$
$$= s^2 \sum_{k=0}^{N} (k+1)s^k(x-v)^k.$$

We will find the difference $\Delta_N^*(x) = y_x'^* - y^{*\,2}$, which is an error – function, i.e.,

$$\Delta_N^*(x) = s^2 \sum_{k=0}^{N-1} (k+1)s^k(x-v)^k - s^2 \sum_{k=0}^{N} (k+1)s^k(x-v)^k$$
$$= (N+1)s^{N+2}(x-v)^N.$$
$$\Delta_N^*(x) \underset{N\to\infty}{\longrightarrow} 0, \quad \text{as } |s||x-v| < 1.$$

The above analysis is applicable if and only if the said solutions are written in their operator form.

Non-direct digital analysis

1. Let the second order nonlinear Mathieu's differential equation

$$y'' + Hy' + \beta^2(1 + a\cos wx)\sin y = 0$$

with the initial conditions $y(x)|_{x=v} = s$, $(y(x))_x'|_{x=v} = t$, be given, [2]; here H, β, a, w are real parameters.

The symbolic expression of the polynomial

$$\hat{y}(x; s; t; v) = \sum_{k=0}^{N} p_k(s,t,v)\frac{(x-v)^k}{k!},$$

where $p_0(s,t,v) = s$;

$$p_{k+1}(s,t,v) = \big(\mathbf{D}_v + t\mathbf{D}_s - \big(Ht + \beta^2(1 + a\cos wv)\sin s\big)\mathbf{D}_t\big)p_k(s,t,v), \quad k \in Z.$$

Then

$$p_1(s,t,v) = t, \quad p_2(s,t,v) = -(Ht + \beta^2(1 + a\cos wv)\sin s),$$
$$p_3(s,t,v) = \beta^2 aw \sin s \sin wv - \beta^2 t \cos s(1 + a\cos wv)$$
$$+ H\big(Ht + \beta^2(1 + a\cos wv)\sin s\big)\sin s \quad \text{and etc.}$$

After we calculate and differentiate the approximation of the solution, we find $\Delta = \max|\Delta(x)|$, where $\Delta(x) = (y^*(x))'' + h(y^*(x))' + \beta^2(1 + a\cos wx)\sin y^*(x)$.

In Table 1 the error estimates and the calculation time T are presented, with $y(0) = 0.5$, $y'(0) = 0.2$, $H = 0.7$, $\beta = 0.9$, $a = 2$, $w = 1$, $x = 0.20$, for different values of N and h.

Table 1

The error estimates and the time expenditure

N	h	Δ	T	N	h	Δ	T	N	h	Δ	T
7	1	0.02	10 s	8	1	0.01	12 s	9	1	0.001	30 s
7	0.5	0.01	10 s	8	0.5	0.005	12 s	9	0.5	0.0005	28 s
10	1	$1 \cdot 10^{-5}$	64 s	11	1	$1 \cdot 10^{-6}$	122 s	12	1	$1 \cdot 10^{-7}$	296 s
10	0.5	$2 \cdot 10^{-5}$	58 s	11	0.5	$1 \cdot 10^{-6}$	116 s	12	0.5	$1 \cdot 10^{-7}$	268 s

2. Let a system of the second order differential equations

$$\begin{cases} x'' - a(y'' \sin y + y'^2 \cos y) + bx' + x - c(1 - dx'^2)x' = 0, \\ y'' - ex'' \sin y + fy' = 0, \end{cases}$$

with initial conditions $x(t)|_{t=v} = s_1$, $x'_t(t)|_{t=v} = t_1$, $y(t)|_{t=v} = s_2$, $y'_t(t)|_{t=v} = t_2$, be given; here a, b, c, d, e, f are real parameters. The above system describes vibrations of dynamical systems.

Solving the system, we obtain polynomial expressions:

$$\hat{x}(t; s_1; t_1, s_2, t_2, v) = \sum_{k=0}^{N} p_k(s_1, t_1, s_2, t_2, v) \frac{(t-v)^k}{k!},$$

$$\hat{y}(t; s_1; t_1, s_2, t_2 v) = \sum_{k=0}^{N} q_k(s_1, t_1, s_2, t_2, v) \frac{(t-v)^k}{k!};$$

here

$$\mathbf{D} = \Bigg(\mathbf{D}_v + t_1 \mathbf{D}_{s_1} + \left(\frac{-a \cos s_2 t_2^2 + bt_1 + s_1 - c\big(1 - dt_1^2\big)t_1 + adft_2 \sin s_2}{ae \sin^2 s_2 - 1}\right)$$
$$\times \mathbf{D}_{t_1} + \mathbf{D}_v + t_2 \mathbf{D}_{s_2}$$
$$+ \left(\frac{aft_2 - e \sin s_2 \big(a \cos s_2 t_2^2 - bt_1 - s_1 + c\big(1 - dt_1^2\big)t_1\big)}{ae \sin^2 s_2 - 1}\right) \mathbf{D}_{t_2}\Bigg).$$

Then, $p_0 = s_1$, $p_{k+1} = \mathbf{D} p_k$, $q_0 = s_2$, $q_{k+1} = \mathbf{D} q_k$.

Choosing numerical values for constants $a = 0.5$, $b = 0.1$, $c = 0.5$, $d = 2.0$, $e = 0.5$, $f = 0.2$ and the initial conditions $x(0) = 0.5$, $x'_t(t)|_{t=0} = 0.5$, $y(0) = 0.5$, $y'_t(t)|_{t=0} = 0.5$, we get the sought-for approximations of the solutions. The error estimates of Δx and Δy, for polynomials of different degrees N, different step values h and the time expenditure T, are presented in Table 2.

Table 2

The error estimates and the time expenditure

N	h	Δx	Δy	T	N	h	Δx	Δy	T
5	0.5	0.042	0.022	6 s	6	0.5	0.012	0.014	15 s
5	0.2	0.021	0.015	5 s	6	0.2	0.005	0.007	14 s
7	0.5	0.001	0.001	72 s	8	0.5	0.0001	0.0001	236 s
7	0.2	0.001	0.001	68 s	8	0.2	0.0001	0.0001	202 s

5. Conclusions

1. In order to get more accurate approximations, one needs to coordinate the order of a polynomial and the step size.
2. It is expedient to improve the computational strategy and to parallelize the algorithms in use.

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Diferencialinių lygčių sprendinių operatorinių išraiškų realizacija

L. Bikulčienė

Šiame darbe aprašomi ir naudojami netiesinių diferencialinių lygčių ir jų sistemų operatoriniai sprendiniai. Pateiktas sprendinių išraiškų kompiuterinės realizacijos algoritmas ir pavyzdžiai. Sprendinių artinių skaičiavimas atliktas *Maple* programine įranga.