Liet. matem. rink., T. 42, spec. nr., 63–69

A limit theorem for zeta-functions of normalized cusp forms

Antanas LAURINČIKAS (VU)
e-mail: antanas.laurincikas@maf.vu.lt

Let

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d, \in \mathbb{Z},\ ad-bc=1 \right\}$$

be the full modular group. A holomorphic on $\Im z > 0$ function $F(z)$ is called a cusp form of weight κ for the full modular group $SL(2,\mathbb{Z})$ if

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

We assume additionally that $F(z)$ is a normalized Hecke's eigenform. Then $F(z)$ has the Fourier series expansion

$$F(z) = \sum_{m=1}^{\infty} c(m) \mathrm{e}^{2\pi imz}, \quad c(1)=1.$$

E. Hecke introduced the zeta-function $\varphi(s,F)$, for $\sigma > \frac{\kappa+1}{2}$, given by absolutely convergent Dirichlet series with coefficients $c(m)$:

$$\varphi(s,F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s}.$$

The function $\varphi(s,F)$ is analytically continuable to an entire function. Moreover, for $\sigma > (\kappa+1)/2$, $\varphi(s,F)$ has the Euler product expansion

$$\varphi(s,F) = \prod_p \left(1-\frac{\alpha(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)}{p^s}\right)^{-1}$$

with

$$c(p) = \alpha(p) + \beta(p).$$

Let

$$\nu_T(\ldots) = \frac{1}{T}\text{meas}\,\{\tau \in [0, T] : \ldots\},$$

where meas $\{A\}$ denotes the Lebesgue measure of the set A, and in place of dots we write a condition satisfied by τ. Denote by $H(D)$ the space of analytic on $D = \{s \in \mathbb{Z}\colon \sigma > \kappa/2\}$ functions equipped with the topology of uniform convergence on compacta. Let γ be the unit circle on $\mathbb{Z}$, and

$$\Omega = \prod_p \gamma_p, \gamma_p = \gamma \quad \text{for all primes } p.$$

Then Ω is a compact topological group, and on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(S)$ stands for the class of Borel sets of the space S) the probability Haar measure m_H exists. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Define on this probability space an $H(D)$-valued random element $\varphi(s, \omega; F)$ by

$$\varphi(s, \omega; F) = \prod_p \left(1 - \frac{\alpha(p)\omega(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\omega(p)}{p^s}\right)^{-1}, \quad \omega \in \Omega,$$

where $\omega(p)$ is the projection of $\omega \in \Omega$ to the coordinate space γ_p. Let P_φ be the distribution of the random element $\varphi(s, \omega; F)$. Then in [1] the following statement was proved.

Theorem A. *The probability measure*

$$\nu_T\big(\varphi(s + i\tau, F) \in A\big), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_φ as $T \to \infty$.

The aim of this note is to present a limit theorem for the function $\varphi(s, F)$ in the space of continuous functions $C(\mathbb{R})$ equipped with the topology of uniform convergence on compacta. Let ω be an arbitrary complex number and $\sigma > (\kappa + 1)/2$. Define a branch of the multi-valued function $\varphi^\omega(s, F)$ by

$$\varphi^w(s, F) = \exp\{w \log \varphi(s, F)\} = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-w} \left(1 - \frac{\beta(p)}{p^s}\right)^{-w}.$$

Hence we deduce that, for $\sigma > (\kappa + 1)/2$,

$$\varphi^w(s, F) = \prod_p \sum_{k=0}^{\infty} \frac{g_w(p^\alpha)}{p^{\alpha s}} = \sum_{m=1}^{\infty} \frac{g_w(m)}{m^s},$$

where

$$g_w(p^k) = \sum_{p^l | p^k} d_w(p^l)\alpha^l(p) d_w(p^{k-l})\beta^{k-l}(p),$$

$$d_w(p^k) = \frac{w(w+1)\dots(w+k-1)}{k!}, \quad k = 1, 2, \dots,$$

and $g_w(m)$ is a multiplicative function.

Let $\theta > \sqrt{2}/2$ be fixed,

$$\sigma_T = \frac{\kappa}{2} + \frac{\theta(\log\log T)^{3/2}}{\log T}, \quad \kappa_T = (2^{-1}\log\log T)^{-1/2}.$$

Similarly as in the case of the Riemann zeta-function [2] it can be proved that

$$\sum_{m \leqslant T} \frac{g_{\kappa_T}(m)\omega(m)}{m^{\sigma_T + it}}, \quad \omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p),$$

for almost all $\omega \in \Omega$ converges uniformly in t on compact subsets of $\mathbb{R}$ to some function $S(t,\omega)$ as $T \to \infty$. Therefore the limit function $S(t,\omega)$ is a $C(\mathbb{R})$-valued random element defined the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by P_S the distribution of the random element $S(t,\omega)$.

Theorem. *Suppose that the function $\varphi(s,F)$ has no zeros in the region $\sigma > \kappa/2$. Then the probability measure*

$$P_T(A) \overset{def}{=} \nu_T\big(\varphi^{\kappa_T}(\sigma_T + it + i\tau) \in A\big), \quad A \in \mathcal{B}(C(\mathbb{R})),$$

converges weakly to P_S as $T \to \infty$.

Here we will give a sketch of the proof of the theorem.

First we consider the Dirichlet polynomial

$$S_u(s) = \sum_{m \leqslant u} \frac{g_{\kappa_T}(m)}{m^s}.$$

Define on $\big(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R}))\big)$ the probability measure

$$P_{T,S_T}(A) = \nu_T\big(S_T(\sigma_T + it + i\tau) \in A\big).$$

Lemma 1. *The probability measure P_{T,S_T} converges weakly to the measure P_S as $T \to \infty$.*

Proof. First we prove the existence of a probability measure P on $(C(\mathbb{R}), \mathcal{B}(C(\mathbb{R})))$ such that the measure P_{T,S_T} converges weakly to P as $T \to \infty$. This is a consequence of the relation

$$\lim_{T\to\infty} \sum_{m \leqslant T} \frac{g_{\kappa_T}(m)\omega(m)}{m^{\sigma_T+it}} = S(t,\omega),$$

which is valid for almost all $\omega \in \Omega$ uniformly in t on compact subsets of $\mathbb{R}$, of properties of the weak convergence and of a limit theorem for the probability measure

$$\nu_T\big((p_1^{-i\tau}, p_2^{-i\tau}, \dots) \in A\big), \quad A \in \mathcal{B}(\Omega),$$

where p_m denotes the mth prime number, see [2].

The sum $S_T(s)$ is too long, and we will change it by a shorter one. Let

$$n_T = T^{\kappa_T/2}, \quad \varepsilon_T = (\log\log T)^{-1}.$$

Lemma 2. *The probability measure $P_{T,S_{n_T}}$ converges weakly to P_S as $T \to \infty$.*

Proof. There exists a sequence $\{K_j\}$ of compact subsets of $\mathbb{R}$ such that

$$\mathbb{R} = \bigcup_{j=1}^{\infty} K_j, \quad K_j \subset K_{j+1},$$

and if K is a compact subset of $\mathbb{R}$, then $K \subseteq K_j$ for some j. Let

$$\varrho_j(f,g) = \sup_{t\in K_j} d\big(f(t), g(t)\big), \quad f,g \in C(\mathbb{R}).$$

Then

$$\varrho(f,g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\varrho_j(f,g)}{1+\varrho_j(f,g)}$$

is a metric in $C(\mathbb{R})$ which induces its topology. Here d is the spheric metric on the Riemann sphere. Let

$$Z_T(it,\tau) = \sum_{n_T < m \leqslant T} \frac{g_{\kappa_T}(m)}{m^{\sigma_T+it+i\tau}}.$$

Then, using the contour integration and the Montgomery–Vaughan theorem, we find for any compact subset K of $\mathbb{R}$ that

$$\nu_T\bigg(\sup_{t\in K} \big|Z_T(it,\tau)\big| \geqslant \varepsilon_T\bigg) = \frac{B\log T}{\varepsilon_T^2} \sum_{n_T < m \leqslant T} \frac{g_{\kappa_T}^2(m)}{m^{\sigma_T 2/\log T}}$$

$$= \frac{B \log T}{\varepsilon_T^2} T^{-\kappa_T} \frac{\theta(\log\log T)^{3/2} - 2}{\log T} \sum_{n_T < m \leqslant T} \frac{d_{2\kappa_T}^2(m)}{m} = \mathrm{o}(1),$$

as $T \to \infty$. This and the definition of the metric ϱ yield

$$\nu_T\Big(\varrho\big(S_T(\sigma + it + i\tau), S_{n_T}(\sigma_T + it + i\tau)\big) \geqslant \varepsilon\Big)$$

$$\leqslant \frac{1}{\varepsilon} \sum_{j=1}^{\infty} 2^{-j} \frac{1}{T} \int_0^T \frac{2 \sup_{s \in K_j} |Z_T(it, \tau)|}{1 + 2 \sup_{t \in K_j} |Z_T(it, \tau)|} \, \mathrm{d}\tau$$

$$= \frac{1}{\varepsilon} \sum_{j=1}^{\infty} 2^{-j} \frac{1}{T} \Bigg(\int\limits_{\substack{0 \\ \sup_{t \in K_j} |Z_T(it,\tau)| \leqslant \varepsilon_T}}^{T} + \int\limits_{\substack{0 \\ \sup_{t \in K_j} |Z_T(it,\tau)| > \varepsilon_T}}^{T} \Bigg) \frac{2 \sup_{s \in K_j} |Z_T(it, \tau)|}{1 + 2 \sup_{t \in K_j} |Z_T(it, \tau)|} \mathrm{d}\tau = \mathrm{o}(1),$$

as $T \to \infty$. Hence in view of the well-known properties of the weak convergence the lemma follows.

Now let

$$g(s) = \varphi^{\kappa_T}(s, F) - S_{n_T}(s).$$

Our next aim is to obtain the following assertion.

Lemma 3. *Let $\varepsilon_T = 1/\log T$, and let K be a compact subset of $\mathbb{R}$. Then*

$$\nu_T\Big(\sup_{t \in K} \big|g(\sigma_T + it + i\tau)\big| \geqslant \varepsilon_T \Big) = \mathrm{o}(1)$$

as $T \to \infty$.

Proof. We apply the moment method developed in [2]. Let

$$K(\sigma) = \int_{-\infty}^{\infty} \big|g(\sigma + it)\big|^2 \omega(t) \, \mathrm{d}t,$$

with

$$\omega(t) = \int_{\log^2 T}^{T/2} \exp\big\{ -2(t - 2\tau)\big\} \, \mathrm{d}\tau.$$

Then similarly as in [2] we obtain that, for $\kappa/2 \leqslant \sigma \leqslant \sigma_2 \leqslant \kappa/2 + 1/16$ and $T \geqslant T_0$,

$$\begin{aligned} K(\sigma_2) &= B\big(K(\sigma_1)\big)^{\frac{4\kappa+3-4\sigma_2-4\sigma_1}{4\kappa+3-8\sigma_1}} \left(T^{1-c_1\kappa_T}\right)^{\frac{4(\sigma_2-\sigma_1)}{4\kappa+3-8\sigma_1}} \\ &\quad + B\big(K(\sigma_1)\big)^{\frac{4\kappa+3-8\sigma_2}{4\kappa+3-8\sigma_1}} \exp\left\{ - c_2(\sigma_2 - \sigma_1)\log^2 T\right\}. \end{aligned} \tag{1}$$

Further, we define

$$L(\sigma) = \int_{-\infty}^{\infty} \left|S_{n_T}(\sigma + it)\right|^{2/\kappa_T} \omega(t)\,\mathrm{d}t$$

and

$$J(\sigma) = \int_{-\infty}^{\infty} \left|\varphi(\sigma + it, F)\right|^2 \omega(t)\,\mathrm{d}t,$$

and prove that, for $T \geqslant T_0$,

$$L\Big(\frac{\kappa}{2} + \frac{1}{\log T}\Big) = BT(\log T)^{1+\varepsilon}, \quad \varepsilon > 0,$$
$$J\Big(\frac{\kappa}{2} + \frac{1}{\log T}\Big) = BT\log^4 T.$$

From this and (1) we deduce that, for $\sigma_T - \frac{1}{\log T} \leqslant \tilde{\sigma}_T \leqslant \sigma_T + \frac{1}{\log T}$ and $T \geqslant T_0$,

$$K(\tilde{\sigma}_T) = BT\exp\left\{ - c_3(\log\log T)^{3/2}\right\}. \tag{2}$$

Now by Chebyshev's inequality, using the contour integration, we find

$$\begin{aligned} \nu_T\Big(\sup_{t\in K} \left|g(\sigma_T + it + i\tau)\right| \geqslant \varepsilon_T\Big) &\leqslant \frac{1}{\varepsilon_T^2 T}\int_0^T \sup_{t\in K} \left|g(\sigma_T + it + i\tau)\right|^2 \mathrm{d}\tau \\ &= B\log T\int_0^{2T} \left|g(\sigma_T + it + i\tau)\right|^2 \mathrm{d}t = BT\exp\left\{ - c_3(\log\log T)^{3/2}\right\}. \end{aligned}$$

This and (2) prove the lemma.

Proof of the Theorem. Let ε be an arbitrary positive number. Then

$$\begin{aligned} &\nu_T\Big(\varrho\big(\varphi^{\kappa_T}(\sigma + it + i\tau), S_{n_T}(\sigma_T + it + i\tau)\big) \geqslant \varepsilon\Big) \\ &\leqslant \frac{1}{\varepsilon}\sum_{j=1}^{\infty} 2^{-j}\frac{1}{T}\int_0^T \frac{2\sup_{t\in K_j} \left|g(\sigma + it + i\tau)\right|}{1 + 2\sup_{t\in K_j} \left|g(\sigma + it + i\tau)\right|}\,\mathrm{d}\tau \end{aligned}$$

$$=\frac{1}{\varepsilon}\sum_{j=1}^{\infty}2^{-j}\frac{1}{T}\Bigg(\int\limits_{\substack{0\\ \sup\limits_{t\in K_j}|g(\sigma+it+i\tau)|<\varepsilon_T}}^{T}+\int\limits_{\substack{0\\ \sup\limits_{t\in K_j}|g(\sigma+it+i\tau)|>\varepsilon_T}}^{T}\Bigg)\frac{2\sup\limits_{t\in K_j}|g(\sigma+it+i\tau)|}{1+2\sup\limits_{t\in K_j}|g(\sigma+it+i\tau)|}\mathrm{d}\tau=\mathrm{o}(1),$$

as $T\to\infty$ by Lemma 3. Hence the theorem is a consequence of Lemma 2.

References

[1] A. Kačėnas, A. Laurinčikas, On Dirichlet series related to certain cusp forms, *Lith. Math. J.*, **38**, 64–76 (1998).

[2] A. Laurinčikas, *Limit Theorems fo the Riemann Zeta-Function*, Kluwer, Dordrecht/London/Boston (1996).

Ribinė teorema normuotų parabolinių formų dzeta funkcijoms

A. Laurinčikas

Straipsnyje įrodyta ribinė teorema silpno matų konvergavimo prasme normuotų parabolinių formų dzeta funkcijoms tolydžių funkcijų erdvėje.