

The support of one random element

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In [2] a joint limit theorem for general Dirichlet series was obtained. Let $s = \sigma + it$ be a complex variable, and let, for $\sigma > \sigma_{aj}$, $f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_{mj}s}$, $j = 1, \dots, n$. Here $\{a_{mj}\}$ is a sequence of complex numbers, and $\{\lambda_{mj}\}$ is an increasing sequence of real numbers, $\lim_{m \rightarrow \infty} \lambda_{mj} = +\infty$, $j = 1, \dots, n$. We assume that the function $f_j(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_{1j}$, $\sigma_{1j} < \sigma_{aj}$, $j = 1, \dots, n$, and all poles in this region are included in a compact set. We also require that, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(s) = O(|t|^{\delta_j}), \quad |t| \geq t_0, \quad \delta_j > 0, \quad (1)$$

and

$$\int_{-T}^T |f_j(\sigma + it)|^2 dt = O(T), \quad T \rightarrow \infty, \quad (2)$$

hold, where $j = 1, \dots, n$. Moreover, we assume that

$$\lambda_{mj} \geq c_j (\log m)^{\theta_j} \quad (3)$$

with some positive constants c_j and θ_j , $j = 1, \dots, n$. Denote by $H(D_j)$, $D_j = \{s \in \mathbb{C} : \sigma > \sigma_{1j}\}$, $j = 1, \dots, n$, the space of analytic on D_j functions equipped with the topology of uniform convergence on compacta, and let $H_n = H_n(D_1, \dots, D_n) = H(D_1) \times \dots \times H(D_n)$. Let $\Omega = \prod_{m=1}^{\infty} \gamma_m$, where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}$. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S . Then we have a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ where m_H is the Haar measure on $(\Omega, \mathcal{B}(\Omega))$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m . Now define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ an H_n -valued random element $F(s_1, \dots, s_n; \omega)$ by the formula

$$F(s_1, \dots, s_n; \omega) = (f_1(s_1, \omega), \dots, f_n(s_n, \omega)),$$

where

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj} \omega(m) e^{-\lambda_{mj}s_j}, \quad s_j \in D_j, \quad j = 1, \dots, n.$$

Let $M(D_j)$ stand for the space of meromorphic on D_j functions equipped with the topology of uniform convergence on compacta, and $M_n = M_n(D_1, \dots, D_n) = M(D_1) \times \dots \times M(D_n)$. In [2] the following limit theorem on M_n was proved.

Theorem 1. For $j = 1, \dots, n$, suppose that the sets $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$ are linearly independent over the field of rational numbers, and that for $f_j(s)$ the conditions (1)–(3) are satisfied. Then the probability measure

$$\frac{1}{T} \text{meas}\{\tau \in [0, T]: (f_1(s_1 + i\tau), \dots, f_n(s_n + i\tau)) \in A\}, \quad A \in \mathcal{B}(M_n),$$

converges weakly to the distribution P_F of the random element $F(s_1, \dots, s_n, \omega)$.

For applications it is useful to know the support of the measure P_F , and the present note is devoted to this problem.

We need some additional conditions on the sequences $\{a_{mj}\}$ and $\{\lambda_{mj}\}$. We suppose that $\lambda_{mj} = \lambda_m$, $j = 1, \dots, n$, and let $c_{mj} = a_{mj}e^{-\lambda_m \sigma_{aj}}$, $j = 1, \dots, n$. Assume that there exist $r \geq n$ sets \mathbb{N}_k , $\mathbb{N}_{k_1} \cap \mathbb{N}_{k_2} = \emptyset$ for $k_1 \neq k_2$, $\mathbb{N} = \bigcup_{k=1}^r \mathbb{N}_k$, such that $c_{mj} = b_{kj}$ for $m \in \mathbb{N}_k$, $k = 1, \dots, r$, $j = 1, \dots, n$. We set

$$\mathcal{B} = \begin{pmatrix} b_{11}, \dots, b_{1n} \\ \dots, \dots, \dots \\ b_{r1}, \dots, b_{rn} \end{pmatrix}.$$

We also suppose that the sequence of exponents $\{\lambda_m\}$ satisfies the relation

$$r(x) = \sum_{\lambda_m \leq x} 1 = C_1 x^\kappa + B, \quad (4)$$

with $\kappa \geq 1$ and some positive C_1 , and $|B| \leq C_2$, and that

$$\sum_{\substack{\lambda_m \leq x \\ m \in \mathbb{N}_k}} 1 = \kappa_k r(x)(1 + o(1)), \quad x \rightarrow \infty, \quad (5)$$

with positive κ_k , $k = 1, \dots, r$.

Let N be an arbitrary positive number, and let $D_{j,N} = \{s \in \mathbb{C}: \sigma_{1j} < \sigma < \sigma_{aj}, |t| < N\}$, $j = 1, \dots, n$. Denote by $\widehat{F}(s_1, \dots, s_n; \omega)$ the restriction of $F(s_1, \dots, s_n, \omega)$ to $H_{n,N} = H(D_{1,N}) \times \dots \times H(D_{n,N})$.

Theorem 2. Suppose that conditions (1), (2) and (4), (5) are satisfied, and that $\text{rank}(\mathcal{B}) = n$. Then the support of the random element $\widehat{F}(s_1, \dots, s_n; \omega)$ is the whole of $H_{n,N}$.

We begin with one lemma on functions of exponential type. Let ν be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in $\{s \in \mathbb{C}: \min_{1 \leq j \leq n} (\sigma_{1j} - \sigma_{aj}) < \sigma < 0, |t| < N\}$. Define

$$w(z) = \int_{\mathbb{C}} e^{-sz} d\nu(s), \quad z \in \mathbb{C}.$$

Lemma 1. *Suppose that, for some k ,*

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} |w(\lambda_m)| < \infty.$$

Then $\int_{\mathbb{C}} s^l d\nu(s) = 0, \quad l = 0, 1, 2, \dots$

Proof . Since, for $x > 0$,

$$|w(\pm x)| \leq e^{Nx} \int_{\mathbb{C}} |d\nu(s)|,$$

condition 1^o of Theorem 6.4.12 from [1] is satisfied with $\alpha = N$. Now we fix β satisfying $0 < \beta < \frac{\pi}{N}$, and a real number ξ with

$$C_1 \beta \xi > C_2, \tag{6}$$

where the constants C_1 and C_2 are given by formula (4). Define $A = \{l \in \mathbb{N}: \exists r \in [(l - \xi)\beta, (l + \xi)\beta] \text{ with } |w(r)| \leq r^{-\kappa}\}$. Then we have

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} |w(\lambda_m)| \geq \sum_{l \notin A} \sum'_m |w(\lambda_m)| \geq \sum_{l \notin A} \sum'_m \lambda_m^{-\kappa},$$

where \sum'_m denotes the sum over all $m \in \mathbb{N}_k$ satisfying the inequalities $(l - \xi)\beta < \lambda \leq (l + \xi)\beta$. This and the hypothesis of the lemma yield

$$\sum_{l \notin A} \sum_{m \in \mathbb{N}_k} \lambda_m^{-\kappa} < \infty \tag{7}$$

with $a = (l - \xi)\beta, b = (l + \xi)\beta$. Summing by parts and applying (4) and (5), we find

$$\sum_{\substack{m \in \mathbb{N}_k \\ a < \lambda_m \leq b}} \lambda_m^{-\kappa} = \frac{1}{b^\kappa} \sum_{\substack{m \in \mathbb{N}_k \\ a < \lambda_m \leq b}} 1 + \kappa \int_a^b \left(\sum_{\substack{m \in \mathbb{N}_k \\ a < \lambda_m \leq u}} 1 \right) \frac{du}{u^{\kappa+1}}$$

$$\begin{aligned} &\geq (r(b) - r(a)) \frac{\kappa_k}{b^\kappa} (1 + o(1)) \\ &\geq \left(\frac{2C_1 \kappa_k \beta^\kappa l^\kappa \kappa \xi}{b^\kappa l} + \frac{B}{l^2} - \frac{2\kappa_k C_2}{b^\kappa} \right) (1 + o(1)) \end{aligned}$$

as $l \rightarrow \infty$. Therefore, (6) and (7) show that

$$\sum_{l \notin A} \frac{1}{l} < \infty. \quad (8)$$

Let $A = \{a_l : l \in \mathbb{N}\}$, $a_1 < a_2 < \dots$. Then from (8) we find

$$\lim_{l \rightarrow \infty} \frac{a_l}{l} = 1. \quad (9)$$

By the definition of the set A there exists a sequence $\{\xi_l\}$ such that $(a_l - \xi)\beta < \xi_l \leq (a_l + \xi)\beta$ and $|w(\xi_l)| \leq \xi_l^{-\kappa}$. This and (9) show that $\lim_{l \rightarrow \infty} \xi_l/l = \beta$, and $\limsup_{l \rightarrow \infty} \frac{\log |w(\xi_l)|}{\xi_l} \leq 0$.

Moreover, in view of (9) $|\xi_m - \xi_n| > |a_m - a_n|\delta \geq \delta|m - n|$ with some positive constant δ . therefore, applying Theorem 6.4.12 [1], we find that

$$\limsup_{r \rightarrow \infty} \frac{\log |w(r)|}{r} \leq 0. \quad (10)$$

On the other hand, by Lemma 6.4.10 [1], if $w(s) \not\equiv 0$, then $\limsup_{r \rightarrow \infty} \frac{\log |w(r)|}{r} > 0$, and this contradicts (10). Consequently, $w(s) \equiv 0$, and the lemma follows by differentiation.

Proof of Theorem 2. Let, for $m = 1, 2, \dots$,

$$\begin{aligned} \underline{f}_m(s_1, \dots, s_n; \omega(m)) &= (f_m(s_1, \omega), \dots, f_n(s_n, \omega)) \\ &= (a_{m1}\omega(m)e^{-\lambda_m s_1}, \dots, a_{mn}\omega(m)e^{-\lambda_m s_n}). \end{aligned}$$

From the definition of Ω it follows that $\{\omega(m)\}$ is a sequence of independent random variables with respect to the measure m_H . Hence $\{\underline{f}_m(s_1, \dots, s_n; \omega(m))\}$ is a sequence of independent $H_{n,N}$ -valued random elements defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. The support of each random variable $\omega(m)$ is the unit circle γ . Therefore, the set $\{\underline{f}_m(s_1, \dots, s_n; a) : a \in \gamma\}$ is the support of the random element $\underline{f}(s_1, \dots, s_n; \omega(m))$. Hence in virtue of Lemma 5 from [3] the closure of the set of all convergent series

$$\sum_{m=1}^{\infty} \underline{f}_m(s_1, \dots, s_n; a_m), \quad a_m \in \gamma,$$

is the support of the random element $\widehat{F}(s_1, \dots, s_n; \omega)$. To prove the theorem it remains to check that the latter set is dense in $H_{n,N}$. For this we will apply Lemma 6 from [3].

Let μ_1, \dots, μ_n be complex measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in $D_{1,N}, \dots, D_{n,N}$, respectively, such that

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^n \int_{\mathbb{C}} a_{mj} e^{-\lambda_m s} d\mu_j(s) \right| < \infty. \quad (11)$$

Now let $h_j(s) = s - \sigma_{aj}, j = 1, \dots, n$. Then we have that $\mu_j h^{-1}(A) = \mu_j(h_j^{-1}A), A \in \mathcal{B}(\mathbb{C})$, is a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in $\widehat{D}_{j,N} = \{s \in \mathbb{C} : \sigma_{1j} < \sigma_{aj} < \sigma < 0, |t| \leq N\}$. Now (11) can be rewritten in the form

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^n c_{mj} \int_{\mathbb{C}} e^{-\lambda_m s} d\mu_j h_j^{-1}(s) \right| < \infty.$$

This together with hypothesis on c_{mj} leads to

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} \left| \sum_{j=1}^n b_{kj} \int_{\mathbb{C}} e^{-\lambda_{mj} s} d\mu_j h_j^{-1}(s) \right| < \infty, \quad k = 1, \dots, r. \quad (12)$$

Taking $\widehat{\mu}_k(A) = \sum_{j=1}^n b_{kj} \mu_j h_j^{-1}(A), A \in \mathcal{B}(\mathbb{C}), v_k(z) = \int_{\mathbb{C}} e^{-sz} d\mu_k(s), z \in \mathbb{C}, k = 1, \dots, r$, we write (12) in the form

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} |w_k(A_m)| < \infty, \quad k = 1, \dots, r.$$

Now Lemma 1 shows that $v_k(z) = 0$, and therefore $\int_{\mathbb{C}} s^l d\widehat{\mu}_k(s) = 0, l = 0, 1, 2, \dots, k = 1, \dots, r$. Hence, using the definition of $\widehat{\mu}_k$ and the properties of the matrix \mathcal{B} , we obtain that $\int_{\mathbb{C}} s^l d\mu_j h_j^{-1}(s) = 0, l = 0, 1, 2, \dots, j = 1, 2, \dots, n$, and this together with definition of the function h_j implies the relations

$$\int_{\mathbb{C}} s^l d\mu_j(s) = 0, \quad l = 0, 1, 2, \dots, \quad j = 1, 2, \dots, n. \quad (13)$$

Moreover, we have [4] that there exists a sequence $\{b_m : b_m \in \gamma\}$ such that $\sum_{m=1}^{\infty} \underline{f}_m(s_1, \dots, s_n; b_m)$ converges in $H_{n,N}$, and $\sum_{m=1}^{\infty} \sum_{j=1}^n \sup_{s \in K_j} |f_{mj}(s, b_m)|^2 < \infty$. Since $|b_m| = 1$, condition (13) remains also valid for $f(s_1, \dots, s_n; b_m)$. Thus we have that all conditions of Lemma 6 [3] are satisfied, and therefore the set of all convergent series

$$\sum_{m=1}^{\infty} a_m^* \underline{f}_m(s_1, \dots, s_n; b_m), \quad a_m^* \in \gamma,$$

is dense in $H_{n,N}$. Hence the set of all convergent series

$$\sum_{m=1}^{\infty} f_m(s_1, \dots, s_n; a_m), \quad a_m \in \gamma,$$

is dense in $H_{n,N}$, and the closure of this set is the whole of $H_{n,N}$. The theorem is proved.

References

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Vieno atsitiktinio elemento atrama

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Nagrinėjamas atsitiktinis elementas, igyjantis reikšmes daugiamatėje analizinių funkcijų erdvėje ir įrodoma, jog šio elemento atrama yra visa minėta erdvė.