Prefixed tableaux for multi-modal logic of knowledge with inclusions

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1. Introduction

The logic denoted by $S5$ and its multi-modal version denoted by $S5_n$ are widely recognised as logics of knowledge (see, e.g., [4]).

In this paper we consider the certain extensions of the logic $S5_n$. $S5_n$ contains an indexed set of n independent unary modal connectives that allow us to represent the information possessed by a group of agents. We consider a family of the extensions of $S5_n$, where these connectives are interrelated, i.e., an accessibility relation can be included into another one. We denote a set of such inclusions by **I** and we denote the corresponding extension of $S5_n$ by $S5_n^I$.

In this paper, we present a prefixed analytic tableau proof system for $S5_n^I$ which is a generalization of the prefixed tableau proof system described by Massacci for $S5$ [5]. We use prefixes reasoning about accessibility. But prefixes are more complex than in the mono-modal case. We prove soundness and completeness of the presented system.

Some prefixed tableau systems for $S5_n^I$ are presented in [1] and [7]. Our system is closer to that described in [7] but we do not use the side conditions on inference rules reasoning about accessibility.

Our paper is organized as follows. In Section 2 we give syntax and semantics of the logic $S5_n^I$. In Section 3 we present a tableau calculus for $S5_n^I$. In Section 4 we sketch the proof of soundness and completeness of this calculus.

2. Syntax and semantics of $S5_n^I$

Given the set $\mathcal{P} = \{p, \ldots\}$ of propositional variables, the set of formulas is inductively defined as follows:

1) any $p \in \mathcal{P}$ is a formula and is called an atomic formula;

2) if A, B are formulas then so are $\neg A, A \wedge B, \square_1 A, \ldots, \square_n A$.

A *frame* is a structure $\mathcal{F} = <W, R_1, \ldots, R_n>$ such that W is a non-empty set, whose elements are called *worlds*, and R_i is a binary relation on $W, i \in \{1, \ldots, n\}$. We write wR_iw' iff $(w, w') \in R_i$, and we say that w' is accessible from w by the relation R_i. A *model* is a structure $\mathcal{M} = <\mathcal{F}, V>$ such that $\mathcal{F}$ is a frame and V is a valuation V: $\mathcal{P} \rightarrow 2^W$. The satisfiability relation $\models$ is defined inductively in the usual way:

$$\mathcal{M}, w \models p \qquad \text{iff} \quad w \in V(p) \text{ for every } p \in \mathcal{P};$$

$$\mathcal{M}, w \models \neg A \qquad \text{iff} \quad \mathcal{M}, w \not\models A;$$

$$\mathcal{M}, w \models A \wedge B \quad \text{iff} \quad \mathcal{M}, w \models A \text{ and } \mathcal{M}, w \models B;$$

$$\mathcal{M}, w \models \Box_i A \qquad \text{iff} \quad \text{for every } v \text{ such that } wR_i v, \text{ we have } \mathcal{M}, v \models A.$$

A formula A is said to be *true* in the model (written $\mathcal{M} \models A$) iff for every world $w \in W$ $\mathcal{M}, w \models A$. A formula A is said to be *true* in the frame $\mathcal{F}$ (written $\mathcal{F} \models A$) iff A is true in every model based on $\mathcal{F}$.

Let $\mathbf{I}$ be a subset of the set $\{R_{i_1} \subseteq R_{i_2} | 1 \leqslant i_1, i_2 \leqslant n\}$ of inclusions such that: 1) $R_i \subseteq R_i \in \mathbf{I}$ for each $1 \leqslant i \leqslant n$; 2) if $R_i \subseteq R_j \in \mathbf{I}$ and $R_j \subseteq R_k \in \mathbf{I}$, then $R_i \subseteq R_k \in \mathbf{I}$.

A *frame of logic* $S5_n^{\mathbf{I}}$ is a frame $< W, R_1, \ldots, R_n >$ such that: 1) for every $1 \leqslant i \leqslant n$ R_i is an equivalence relation; 2) $R_1, \ldots, R_n$ satisfy the property $\mathbf{I}$. A formula A is said to be $S5_n^{\mathbf{I}}$-*satisfiable* iff there exists a model $\mathcal{M}$ based on some frame of the logic $S5_n^{\mathbf{I}}$ and $w \in W$ such that $\mathcal{M}, w \models A$. A formula A is said to be $S5_n^{\mathbf{I}}$-*valid* iff for every frame $\mathcal{F}$ of the logic $S5_n^{\mathbf{I}}$, we have $\mathcal{F} \models A$.

3. Tableau calculus for $S5_n^{\mathbf{I}}$

By $\mathcal{I}, \mathcal{J}$ below we denote subsets of $\{1, \ldots, n\}$. A *prefix* is an alternating sequence of integers and subsets of $\{1, \ldots, n\}$. (This is a generalization of modal case (e.g., [2])).

DEFINITION 3.2. The set of prefixes Σ is the least set such that $1 \in \Sigma$ and if $\sigma \in \Sigma$, $\mathcal{I} \subseteq \{1, \ldots, n\}$ and m is an integer, then $\sigma \mathcal{I} m \in \Sigma$.

Intuitively a prefix $\sigma\{i_1, \ldots, i_k\}m$ names a world accessible by relations $R_{i_1}, \ldots, R_{i_k}$ from a world named by σ.

Tableaus use *prefixed formulas*, i.e., pairs $\sigma: A$ where σ is a prefix and A is a formula. A *tableau* $\mathcal{T}$ is a binary tree each node of which is labelled with a prefixed formula. A *branch* $\mathcal{B}$ is a path from the root downwards in such a tree. Nodes are added and labelled by applying the tableaux rules presented below. The rules are applied downwards as defined for prefixed tableaux by Fitting [2]. A prefix σ is *present* in a branch if there is a prefixed formula $\sigma: A$ for some formula A in that branch. A prefix $\sigma \mathcal{I} n$ is *new* to a branch if there is no prefix of a kind $\sigma \mathcal{J} n$ present in that branch.

The tableau calculus for $S5_n^{\mathbf{I}}$ consists of the following tableaux rules.

1. Propositional connective tableaux rules:

$$\frac{\sigma: A \wedge B}{\begin{array}{c} \sigma: A \\ \sigma: B \end{array}} (\wedge) \qquad \frac{\sigma: \neg(A \wedge B)}{\sigma: \neg A \,|\, \sigma: \neg B} (\vee) \qquad \frac{\sigma: \neg\neg A}{\sigma: A} (\neg)$$

2. Modal tableaux rules. Below we fix $i \in \{1, \ldots, n\}$.

2.1. π_i-rule.

Let $R_i \subseteq R_{i_1}, \ldots, R_i \subseteq R_{i_k}$ be all inclusions from $\mathbf{I}$ where R_i is on the left side of an inclusion.

$$\frac{\sigma: \neg\Box_i A}{\sigma\{i, i_1, \ldots, i_k\}n: \neg A} \quad (\pi_i), \text{ where } \sigma\{i, i_1, \ldots, i_k\}n \text{ is new to the current branch.}$$

2.2. ν_i-rules:

$$\frac{\sigma: \Box_i A}{\sigma\mathcal{I}n: A} (K_i) \qquad \frac{\sigma: \Box_i A}{\sigma: A} (T_i) \qquad \frac{\sigma: \Box_i A}{\sigma\mathcal{I}n: \Box_i A} (4_i) \qquad \frac{\sigma\mathcal{I}n: \Box_i A}{\sigma: \Box_i A} (4^r{}_i)$$

Here $\sigma, \sigma\mathcal{I}n$ must be present in the branch and $i \in \mathcal{I}$.

A branch $\mathcal{B}$ is *closed* if it contains some prefixed formula $\sigma: A$ and also $\sigma: \neg A$. A tableau is *closed* if every branch of it is closed, otherwise it is open. A *tableau proof* for a formula A is a closed tableau starting from $1: \neg A$.

4. Soundness and completeness

Theorem 4.1 (soundness). *If a formula A has a tableau proof, then A is a $S5_n^{\mathbf{I}}$-valid formula.*

The proof of the theorem is similar to that presented in [6].

The rest of the section is devoted to the sketch of the proof of the following

Theorem 4.2 (completeness). *If a formula A is a $S5_n^{\mathbf{I}}$-valid formula, then A has a tableau proof.*

The completeness proof follows the ideas of Fitting [2]: a systematic tableau construction is presented that must produce a tableau proof if one exists and, if one does not, it will give us the information necessary to construct a counter-model. The counter-model is built from saturated open branches (see definitions below).

For calculus for $S5_n^{\mathbf{I}}$ we get a systematic tableau construction by adapting one presented by Gore in [3], page 374: instead of using π-, ν- rules of the mono-modal case use π_i-, ν_i- rules $(i \in \{1, \ldots, n\})$, respectively, of the calculus for $S5_n^{\mathbf{I}}$.

In order to prove the model existence theorem (see Theorem 4.9 below) which is the basis for the proof of the completeness theorem we need some preliminary definitions and lemmas.

Let $\mathcal{B}$ be a branch. We introduce the binary relations $\rhd_i$, $i \in \{1, \ldots, n\}$ over prefixes present in $\mathcal{B}$. $\rhd_i$ is defined as follows: 1) $\sigma \rhd_i \sigma$; 2) $\sigma \rhd_i \sigma\mathcal{I}_1 m_1 \ldots \mathcal{I}_k m_k$, $k \geq 1$, $i \in \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_k$; 3)$\sigma\mathcal{I}_1 m_1 \ldots \mathcal{I}_k m_k \rhd_i \sigma$, $k \geq 1$, $i \in \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_k$; 4)$\sigma\mathcal{I}_1 m_1 \ldots \mathcal{I}_k m_k \rhd_i \sigma\mathcal{J}_1 n_1 \ldots \mathcal{J}_l n_l$, $k \geq 1$, $l \geq 1$, $i \in \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_k$, $i \in \mathcal{J}_1 \cap \ldots \cap \mathcal{J}_l$. Here σ is a non-empty sequence.

It can be verified

Lemma 4.3. 1) $\triangleright_i$ *is an equivalence relation.* 2) *If* $R_i \subseteq R_j \in \mathbf{I}$, *then* $\triangleright_i \subseteq \triangleright_j$.

Let $\mathcal{B}$ be a branch of a tableau.

DEFINITION 4.4. A prefixed formula $\sigma\colon A$ is *reduced* by the rule (ρ) in $\mathcal{B}$

- if (ρ) has the form $\dfrac{\sigma\colon A}{\sigma_1\colon A_1}$, then $\sigma_1\colon A_1$ is in $\mathcal{B}$;

- if (ρ) has the form $\dfrac{\sigma\colon A}{\sigma_1\colon A_1}$, then $\sigma_1\colon A_1$ and $\sigma_2\colon A_2$ are in $\mathcal{B}$;
  $$\sigma_2\colon A_2$$

- if (ρ) has the form $\dfrac{\sigma\colon A}{\sigma_1\colon A_1 \mid \sigma_2\colon A_2}$, then at least one of $\sigma_1\colon A_1$ and $\sigma_2\colon A_2$ is in $\mathcal{B}$.

The formula $\sigma\colon A$ is *fully reduced* in $\mathcal{B}$ if it is reduced by all applicable rules. A prefix σ is *fully reduced* if all prefixed formula $\sigma\colon A$ are fully reduced.

DEFINITION 4.5. A branch $\mathcal{B}$ is *saturated* if all prefixes in $\mathcal{B}$ are fully reduced.

Similarly as in [6] we can prove

Lemma 4.6. *Let $\mathcal{B}$ be a branch of a tableau. If $\mathcal{B}$ is saturated, then the following holds: if $\sigma\colon \Box_i A$ is in $\mathcal{B}$, σ^* is present in $\mathcal{B}$, and $\sigma \triangleright_i \sigma^*$, then $\sigma^*\colon A$ is also in $\mathcal{B}$.*

Now we define a mapping between names and worlds and introduce the notion of satisfiable branch.

DEFINITION 4.7. Let $\mathcal{B}$ be a branch. Let $<W, R_1, \ldots, R_n, V>$ be a model, an *interpretation* $\imath()$ in the model is a mapping from the set of prefixes that occur in $\mathcal{B}$ to worlds W such that for all σ and $\sigma\mathcal{I}n$ present in $\mathcal{B}$ one has $\imath(\sigma)R_j\imath(\sigma\mathcal{I}n)$ for each $j \in \mathcal{I}$.

DEFINITION 4.8. A tableau branch $\mathcal{B}$ is satisfiable iff there is a model $< W, R_1, \ldots, R_n, V >$ and an interpretation $\imath()$ in this model such that for every prefixed formula $\sigma\colon A$ in $\mathcal{B}$, $\imath(\sigma) \models A$.

Theorem 4.9 (model existence). *If $\mathcal{B}$ is a saturated and an open branch, then there exists a model where $\mathcal{B}$ is satisfiable.*

Proof. Construct the model as follows:

$W = \{\sigma \mid \sigma$ is present in $\mathcal{B}\}$;

$\sigma R_i \sigma^*$ iff $\sigma \triangleright_i \sigma^*$, $\quad 1 \leqslant i \leqslant n$;

$V(p) = \{\sigma \mid \sigma\colon p \in \mathcal{B}\}$.

By Lemma 4.2 $< W, R_1, \ldots, R_n >$ is a frame we are looking for. We must show that B is satisfiable in the model $< W, R_1, \ldots, R_n, V >$ with interpretation $\imath()$ (here $\imath()$ is the identity function): for every A we must prove $\sigma \colon A \in B$ then $\sigma \models A$ by induction on the construction of A.

We consider only modal connectives. Let $\sigma \colon \neg \square_i A \in B$. Then we have $\sigma \mathcal{I} n \colon \neg A \in B$ for some $\sigma \mathcal{I} n$, where $i \in \mathcal{I}$ (since $\sigma \colon \neg \square_i A$ must be reduced by the rule (π_i)). By inductive hypothesis $\sigma \mathcal{I} n \models \neg A$ and $\sigma R_i \sigma \mathcal{I} n$ by construction. Therefore $\sigma \models \neg \square_i A$. Let $\sigma \colon \square_i A \in B$. Then by Lemma 4.5, for every σ^* present in B such that $\sigma \rhd_i \sigma^*$, one has $\sigma^* \colon A \in B$. By inductive hypothesis $\sigma^* \models A$ and $\sigma R_i \sigma^*$ by construction. Therefore $\sigma \models \square_i A$.

The proof of completeness Theorem 4.2 is now standard (see page 410 in [2]).

References

[1] M. Baldoni, Normal multimodal logics with interaction axioms, in: D. Basin *et al.* (Eds.), *Labelled Deduction*, 17, Applied Logic Series, Kluwer Academic Publishers (2000), pp. 33–53.

[2] M. Fitting, *Proof Methods for Modal and Intuitionistic Logics*, 169, Synthese Library, D. Reidel Publishing Company, Dordrecht (1983).

[3] R. Goré, Tableau methods for modal and temporal logics, in: D'Agostino *et al.* (Eds.), *Handbook of Tableau Methods*, Kluwer Academic Publishers (1999), pp. 297–396.

[4] J. Halpern, Y. Moses, A guide to completeness and complexity for modal logics of knowledge and belief, *Artificial Intelligence*, 54, 319–379 (1992).

[5] F. Massacci, Strongly analytic tableaux for normal modal logics, in: *Proc. 12th Conference on Automated Deduction*, LNAI, 814 (1994), pp. 123–137.

[6] F. Massacci, Single step tableaux for modal logics: methodology, computations, algorithms, *Journal of Automated Reasoning*, 24, 319–364 (2000).

[7] M. Stone, *First-Order Multi-Modal Deduction*, Technical Report RUCCS TR 55, Rutgers University Center for Cognitive Science (1999).

Semantinės lentelės su prefiksais žinojimo logikos praplėtimui

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Darbe pateikiamas semantinių lentelių su prefiksais skaičiavimas daugelio modalumų žinojimo logikos tam tikram praplėtimui. Įrodomas šio skaičiavimo neprieštaringumas bei pilnumas.