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Combination of temporal logic with modal logic KD

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1. Introduction

Combinations of modal (including temporal) logics are used as a formal theory that can be helpful for the specification, development, and even the execution of digital agents [4], [5]. Propositional modal and temporal logics are often insufficient for more complex real world situations. First-order modal and temporal logics might be necessary. It is well-known that first-order linear temporal logic, FTL, is incomplete, in general, but it becomes complete after adding an ω-type rule [1]. The analogous situation one can see in the case of a first-order linear temporal logic extended with a modal logic. In [3] a decision procedure for so-called miniscoped fragment of first-order linear temporal logic (FTL) is presented.

The aim of this paper is to present a decision procedure for miniscoped fragment of FTL extended by multi-modal logic KD [4].

2. Infinitary sequent calculus MDG_ω

A language under consideration is obtained from a traditional language of FTL with operators $\bigcirc$ (Next) and $\Box$ (Always) by adding deontic modal operators D_k, where $k \in \{1, \ldots, n\}$. It is assumed that all predicate symbols are flexible (i.e., their value change in time) and constants and function symbols are rigid (i.e., with time-independent meanings). A term and formula are defined as usual. We assume a set of agents $Ag = \{1, \ldots, n\}$ and a formula of the shape $D_k A$ is read as "agent k desires A". The modal operators D_k satisfy analogues of the axioms of the multi-modal logic KD [4], [5].

For simplicity we don't consider intension operators I_k ($k \in \{1, \ldots, n\}$) which also satisfy analogues of the axioms of the multi-modal logic KD. Thus, we consider a linear fragment of the logic BDI from [4], [5] with temporal operators $\bigcirc$, $\Box$ and deontic operators D_k. In [4] decidability of a propositional linear BDI was proved. Here a decision procedure for miniscoped first-order fragment of considered logic is presented.

Let us remember the notions of positive and negative occurrences.

A formula (or some symbol) occurs *positively* in some other formula B if it appears within the scope of no negation signs or in the scope of an even number of negation signs, once all occurrences of $A \supset C$ have been replaced by $\neg A \vee C$; in the opposite case, the formula (symbol) occurs *negatively* in B. For a sequent $S = A_1, \ldots, A_n \to B_1, \ldots, B_m$ positive and negative occurrences are determined just like for the formula $\bigwedge_{i=1}^{n} A_i \supset \bigvee_{i=1}^{m} B_i$. For example, in $\forall x \Box P(x) \to \Box \forall x P(x)$ the first (from the left) occurrences of the symbols $\Box$, $\forall x$ are negative, the second occurrences of the same symbols are positive.

A sequent S is *a miniscoped sequent* if all negative (positive) occurrences of $\forall$ ($\exists$, correspondingly) in S occur only in formulas of the shape $Q\bar{x}E(\bar{x})$ (where $Q \in \{\forall, \exists\}$, $\bar{x} = x_1, \ldots, x_n$, $n \geqslant 0$, E is a predicate symbol). This formula is called a *quasi-atomic formula*; if $Q\bar{x} = \varnothing$, then a quasi-atomic formula becomes an atomic one. A miniscoped sequent S is *temporal-free* if S does not contain temporal operators.

For simplicity we consider so-called Horn-type miniscoped sequents (HM-sequent). A miniscoped sequent S is a *HM-sequent* if S satisfies the following conditions: (a) the sequent S contains only one positive occurrence of an operator σ , where $\sigma \in \{\Box, D_i\}$ (*Horn-type condition*); (b) if a formula $\Box A$ occurs negatively in S then A does not contain positive occurrences of the operator σ^* , where $\sigma^* \in \{\bigcirc, \Box, D_i\}$ (*regularity condition*). A HM-sequent S is an *induction-free HM-sequent*, if S does not contain positive occurrences of $\Box$. Otherwise a HM-sequent S is a *non-induction-free* one.

Let us introduce an infinitary calculus for HM-sequents.

A calculus MDG_ω is defined by the following postulates:

Axioms:

$$\Gamma,\ E(t_1, \ldots, t_n) \to \Delta,\ E(t_1, \ldots, t_n);$$
$$\Gamma,\ E(t_1, \ldots, t_n) \to \Delta,\ \exists x_1 \ldots x_n E(x_1, \ldots, x_n);$$
$$\Gamma,\ \forall x_1 \ldots x_n E(x_1, \ldots, x_n) \to \Delta, E(t_1, \ldots, t_n);$$
$$\Gamma, \forall x_1 \ldots x_n E(t_1(x_1), \ldots, t_n(x_n)) \to \Delta, \exists y_1 \ldots y_n E(p_1(y_1), \ldots, p_n(y_n)),$$

where E is a predicate symbol; $\forall i$ $(1 \leqslant i \leqslant n)$ terms $t_i(x_i)$ and $p_i(y_i)$ are unifiable.

Rules:

1) logical rules consist of traditional invertible rules for logical operators, except the rules $(\forall \to)$, $(\to \exists)$;

2) temporal and modal rules:

$$\frac{\Gamma \to A^0}{\Sigma_1, \bigcirc\Gamma \to \Sigma_2, \bigcirc A^0}(\bigcirc) \qquad \frac{A, \bigcirc\Box A,\ \Gamma \to \Delta}{\Box A,\ \Gamma \to \Delta}(\Box \to)$$

$$\frac{\Gamma \to \Delta, A; \ldots;\ \Gamma \to \Delta, \bigcirc^k A, \ldots}{\Gamma \to \Delta, \Box A}(\to \Box_\omega), \qquad \text{where} \quad k \in \omega;$$

$$\frac{\Gamma^* \to A^0}{\Sigma_1, D_k\Gamma \to \Sigma_2, D_j A^0}(D),$$

where $A^0 \in \{A, \varnothing\}$; if $A^0 = \varnothing$ then $\Gamma^* = \Gamma$, otherwise, i.e., if $A^0 = A$ then Γ^* is a subset of Γ such that $D_k = D_j$;

A calculus MDG is obtained from the calculus MDG_ω by dropping the rule $(\to \Box_\omega)$. *A calculus MKD* is obtained from the calculus MDG by dropping the rule $(\Box \to)$.

Theorem 1 (soundness and ω-completeness of MDG_ω). *Let S be a HM-sequent, then $\forall M \models S \iff MDG_\omega \vdash S$.*

Proof 2. * Using Schütte method, analogously as in [1].

Lemma 1. *The calculus MKD is decidable for the class of temporal-free HM-sequents.*

Now we introduce some canonical forms of HM-sequents.

A HM-sequent S is a primary HM-sequent, if $S = \Sigma_1, D_i\Gamma, \bigcirc\Pi, \Box\Omega \to \Sigma_2, A^0$, where $A^0 = \varnothing$ or A is formula of the following shape D_jB, or $\bigcirc B$, or $\Box B$. For every l $(l \in \{1, 2\})$ $\Sigma_l = \varnothing$ or consists of quasi-atomic formulas; $D_i\Gamma = \varnothing$ or consists of HM-formulas of the shape D_iA ; $\bigcirc\Pi = \varnothing$ or consists of HM-formulas of the shape $\bigcirc A$, where A may contain $\Box$; $\Box\Omega = \varnothing$ or consists of HM-formulas of the shape $\Box A$. A HM-sequent S is a reduced primary HM-sequent if S is a primary one such that $\Box\Omega = \varnothing$ and $A^0 \neq \Box B$.

Now we define rules by which the reduction of an HM-sequent S to a set of primary and reduced primary HM-sequents is carried out.

The following rules are called *reduction* ones (all these rules are applied in the bottom-up manner):

1) logical rules of the calculus MDG, except of $(\forall \to)$, $(\to \exists)$;

2) the temporal rule $(\Box \to)$ of the calculus MDG and the following temporal rule:

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, \bigcirc\Box A}{\Gamma \to \Delta, \Box A}(\to \bigcirc\Box).$$

Lemma 2 (reduction of HM-sequent S to a set of primary and reduced primary HM-sequents). *Let S be a HM-sequent. Then using reduction rules one can automatically construct a reduction of S to a set $\{S_1, \dots, S_n\}$, where S_j $(1 \leqslant j \leqslant n)$ is a primary (reduced primary) HM-sequent; moreover, $MDG_\omega \vdash S \Rightarrow MDG_\omega \vdash S_j$, $j \in \{1, \dots, n\}$.*

3. Decision procedure for HM-sequents

First, let us introduce the following separation rules (SR_i). The rules (SR_i) are bottom-up applied to a reduced primary HM-sequent and have the following shape:

$$\frac{S_i}{\Sigma_1, D_i\Gamma, \bigcirc\Pi \to \Sigma_2, A^0}(SR_i),$$

where $1 \leqslant i \leqslant 3$ and $S_1 = \Sigma_1 \to \Sigma_2$; if $A^0 = \varnothing$ then $S_2 = \Gamma \to$; $S_2 = \Gamma^* \to B$, if $A^0 = D_jB$, where Γ^* is a subset of Γ such that $D_i = D_j$; $S_3 = \Pi \to B$, if $A^0 = \bigcirc B$ and $S_3 = \Pi \to$, if $A^0 = \varnothing$.

Lemma 3 (disjunctive invertibility of (SR_i)). *(a) Let S be a conclusion of (SR_i), and $S_i,\ (i \in \{1,2,3\})$ be a premise of (SR_i). Then if $MDG_\omega \vdash S$ then either* (1) $\Sigma_1 \to \Sigma_2$ *is an axiom of MDG_ω , or (2) $MDG_\omega \vdash S_2$, or $MDG_\omega \vdash S_3$. (b)The choice of cases* (1) *or* (2) *is deterministic.*

A calculus MDG^+ is obtained from the calculus MDG by replacing the rules $(\bigcirc)$, (D) by the rules (SR_i).

Lemma 4. *Let S be an induction-free HM-sequent, then $MDG \vdash S \iff MDG^+ \vdash S$.*

We say that two *formulas A* and A^* *are parametrically identical* (in symbols: $A \approx A^*$) if either $A = A^*$ or A, A^* are congruent, or A, A^* differ only by the corresponding occurrences of eigen-constants of the rules $(\to \forall)$, $(\exists \to)$. We say that *HM-sequents S_i* and S_j *are parametrically identical* (in symbols: $S_i \approx S_j$) if S_i, S_j consist of parametrically identical formulas. We say that a *sequent $S_i = \Gamma \to \Delta$ subsumes a sequent $S_j = \Pi, \Gamma' \to \Delta', \Theta$* (in symbols $S_i \succeq S_j$) if $\Gamma \to \Delta \approx \Gamma' \to \Delta'$.

Let S be HM-sequent and A be a formula from S. The notion subformulas of a formula A ($RSub(A)$) is defined as usual except of two points: (1) if A is a quasi-atomic formula then $RSub(A) = \varnothing$; (2) $RSub(QxB(x))) = RSub(B(c))$, where c is a new variable, Q is $\forall(\exists)$ and occurs positively (negatively) in S. The notion of subformulas of a sequent $S = A_1, \ldots, A_n \to A_{n+1}, \ldots, A_{n+m}$ is defined as $RSub(S) = \bigcup\limits_{i=1}^{n+m} RSub(A_i)$. $R^*Sub(S)$ is a set obtained from $RSub(S)$ by merging parametrically identical formulas. It is obvious that $R^*Sub(S)$ is finite.

Lemma 5. *Let S be an induction-free HM-sequent containing at least one negative occurrence of $\Box$. Then bottom-up applying the rules of calculus MDG^+ we can automatically get deduction tree D such that either each leaf of D is an axiom (in this case $MDG^+ \vdash S$), or there exists a branch of D containing two HM-sequents S^*, S^{**} such that $S^* \succeq S^{**}$ (S^* is called saturated HM-sequent). In this case $MDG^+ \nvdash S$. Therefore the calculus MDG^+ is decidable for induction-free HM-sequents.*

Automatic way of construction of the derivation D and correctness (i.e., preservation of derivability) follows from invertibility of the rules of the calculus MDG^+; termination follows from finiteness of the set $R^*Sub(S)$.

As in [3] the notions of the calculus and deduction-based decision procedure are coincidental.

A calculus $HMSat$ is obtained from the calculus MDG^+ by adding the rule $(\to \bigcirc\Box)$ and a procedure for searching saturated HM-sequents. This procedure reflects an inductive nature of the miniscoped fragment of FTL containing a positive occurrence of $\Box$ [6].

Lemma 6. *Let S be a non-induction-free HM-sequent and D be a deduction tree constructed bottom-up applying the rules of calculus $HMSat$. If each leaf of D is either an* *axiom or a saturated non-induction-free HM-sequent S^* then $HMSat \vdash S$. Otherwise $HMSat \nvdash S$. The deduction tree D is constructed automatically. Therefore the calculus $HMSat$ is decidable.*

This Lemma is justified analogously to Lemma 5

Analogously as in [2] we get

Theorem 2. *Let S be HM-sequent. Then $MDG_\omega \vdash S \iff HMSat \vdash S$.*

From Lemmas 1, 5, 6 and Theorem 2 we get

Theorem 3. *The class of HM-sequents is a decision class; the procedure $HMSat$ is sound and complete for the class of HM-sequents.*

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Laiko logikos ir modalumo logikos KD apjungimas

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Pasiūlyta išprendžiamoji procedūra pirmos eilės tiesinio laiko logikos išplėtimo modalumo logika KD fragmentui. Pasiūlyta išprendžiamoji procedūra yra korektiška ir pilna.