

# On the mean square of the periodic zeta-function

Audrius KAČĖNAS\* (VU), Darius ŠIAUČIŪNAS (VU, ŠU)

*e-mail:* darius.siauciunas@centras.lt

Let  $s = \sigma + it$  be a complex variable, and let  $\mathbf{a} = \{a_m : m \in \mathbb{Z}\}$  be a periodic with period  $k$  sequence of complex numbers. The periodic zeta-function  $\zeta(s; \mathbf{a})$  is defined, for  $\sigma > 1$ , by

$$\zeta(s; \mathbf{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and the equality

$$\zeta(s; \mathbf{a}) = \frac{1}{k^s} \sum_{q=1}^k a_q \zeta\left(s, \frac{q}{k}\right),$$

where  $\zeta(s, \alpha)$  denotes the Hurwitz zeta-function, gives an analytic continuation to the whole complex plane for  $\zeta(s; \mathbf{a})$ . It is regular everywhere except, maybe, for a simple pole at  $s = 1$ .

This note is a continuation of papers [2], [3], and [5], where the asymptotics for the mean square

$$I_T(\sigma) = \int_0^T |\zeta(\sigma + it; \mathbf{a})|^2 dt, \quad \frac{1}{2} \leq \sigma < 1,$$

as  $T \rightarrow \infty$  was studied. Our aim is to obtain, the asymptotics for  $I_T(\sigma_T)$  for  $\sigma_T \rightarrow 1/2 + 0$ , as  $T \rightarrow \infty$ . Let  $\sigma_T = 1/2 + l_T^{-1}$  where  $l_T \rightarrow 0$  and  $l_T \rightarrow \infty$  as  $T \rightarrow \infty$ . Define

$$K = K(k) = \sum_{q=1}^k |a_q|^2,$$

and denote by  $B$  a quantity bounded by a constant. Then we have the following statements.

**Theorem 1.** *Let  $l_T = o(\log T)$  as  $T \rightarrow \infty$ . Then*

$$\begin{aligned} I_T(\sigma_T) &= k^{-2\sigma_T} K T l_T + T \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} + B k^{1-2\sigma_T} K T l_T \exp\left\{-\frac{\log T}{l_T}\right\} \\ &\quad + B k^{-2\sigma_T} K T + B k^{1-\sigma_T} K T^{1/2} \log T. \end{aligned}$$

---

\*Partially supported by Lithuanian Foundation of Studies and Science

**Theorem 2.** Let  $\log T = o(l_T)$  as  $T \rightarrow \infty$ . Then

$$I_T(\sigma_T) = k^{-2\sigma_T} KT \log T + T \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} + Bk^{1-2\sigma_T} KT \\ + Bk^{-2\sigma_T} K \frac{T \log^2 T}{l_T} + Bk^{1-\sigma_T} KT^{1/2} \log T.$$

**Theorem 3.** Let  $\lim_{T \rightarrow \infty} l_T / \log T = \kappa \neq 0$ . Then

$$I_T(\sigma_T) = k^{-2\sigma_T} K \frac{\kappa}{2} (1 - e^{-2/\kappa}) T \log T + T \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} + o(k^{-2\sigma_T} KT \log T) \\ + Bk^{1-2\sigma_T} KT + Bk^{1-\sigma_T} KT^{1/2} \log T$$

as  $T \rightarrow \infty$ .

Proof of Theorems 1–3 is based on the approximate functional equation for the function  $\zeta(s; \mathbf{a})$  obtained in [3]. We also preserve the notation of [3]. The mentioned functional equation gives

$$\int_{T/2}^T |\zeta(\sigma_T + it; \mathbf{a})|^2 dt = k^{-2\sigma_T} \int_{T/2}^T \left| \sum_{q=1}^k a_q \sum_{0 \leq m \leq r} \frac{1}{(m + q/k)^{\sigma_T + it}} \right|^2 dt \\ + k^{-2\sigma_T} \int_{T/2}^T \left( \frac{2\pi}{t} \right)^{2\sigma_T - 1} \left| \sum_{q=1}^k a_q \sum_{1 \leq m \leq n} e^{-2\pi i m \frac{q}{k}} \frac{1}{m^{1-\sigma_T - it}} \right|^2 dt \\ + Bk^{-2\sigma_T} \left| \sum_{q=1}^k a_q \right|^2 \int_{T/2}^T t^{-\sigma_T} dt + Bk^{1-2\sigma_T} K \int_{T/2}^T t^{\sigma_T - 2} dt \\ + Bk^{-2\sigma_T} \left| \int_{T/2}^T \left( \frac{2\pi}{t} \right)^{\sigma_T - 1/2 + it} e^{-it} \sum_{q_1=1}^k a_{q_1} \sum_{q_1=1}^k \bar{a}_{q_2} \right. \\ \times \left. \sum_{0 \leq m_1 \leq r} \frac{1}{(m_1 + q_1/k)^{\sigma_T + it}} \sum_{1 \leq m_2 \leq n} e^{2\pi i m_2 \frac{q_2}{k}} \frac{1}{m_2^{1-\sigma_T + it}} dt \right| \\ + Bk^{-2\sigma_T} \left| \int_{T/2}^T \left( \frac{2\pi}{t} \right)^{\sigma_T/2} \sum_{q_1=1}^k \bar{a}_{q_1} \sum_{q_2=1}^k a_{q_2} \right. \\ \times \left. \sum_{0 \leq m \leq r} \frac{1}{(m + q_1/k)^{\sigma_T - it}} \exp \left\{ i f \left( \frac{q_2}{k}, t \right) \right\} \psi \left( 2y - 2n + l - \frac{q_2}{k} \right) dt \right| \\ + Bk^{-2\sigma_T} \left| \int_{T/2}^T \left( \frac{2\pi}{t} \right)^{3\sigma_T/2 - 1/2 - it} e^{-it} \sum_{q_1=1}^k \bar{a}_{q_1} \sum_{q_2=1}^k a_{q_2} \right.$$

$$\begin{aligned}
& \times \sum_{1 \leq m \leq n} e^{2\pi i m \frac{q_1}{k}} \frac{1}{m^{1-\sigma_T+it}} \exp \left\{ i f \left( \frac{q_2}{k}, t \right) \right\} \psi \left( 2y - 2n + l - \frac{q_2}{2} \right) dt \Big| \\
& + Bk^{-2\sigma_T} \left| \int_{T/2}^T \sum_{q=1}^k a_q \sum_{0 \leq m \leq r} \frac{1}{(m+q/k)^{\sigma_T+it}} \overline{R(\sigma_T+it, k)} dt \right| \\
& + Bk^{-2\sigma_T} \left| \int_{T/2}^T \left( \frac{2\pi}{t} \right)^{\sigma_T-1/2+it} \sum_{q=1}^k a_q \sum_{1 \leq m \leq n} e^{-2\pi i m \frac{q}{k}} \frac{1}{m^{1-\sigma_T-it}} \overline{R(\sigma_T+it, k)} dt \right| \\
& + Bk^{-\sigma_T} \left| \int_{T/2}^T \left( \frac{2\pi}{t} \right)^{\sigma_T/2} \sum_{q=1}^k a_q e^{if(q/k, t)} \psi \left( 2y - 2n - l - \frac{q}{k} \right) \overline{R(\sigma_T+it, k)} dt \right| \\
& \stackrel{\text{def}}{=} \sum_{j=1}^{10} I_j. \tag{1}
\end{aligned}$$

*Proof of Theorem 1.* Let  $T_1 = \max(T/2, 2\pi(m_1 + q_1/k), 2\pi(m_2 + q_1/k)^2)$  and  $M_q(T) = [(T/2\pi)^{1/2} - q/k]$ . Then we have that

$$\begin{aligned}
I_1 &= k^{-2\sigma_T} \sum_{q=1}^k |a_q|^2 \sum_{0 \leq m \leq M_q(T)} \int_{T_1}^T \frac{dt}{(m+q/k)^{2\sigma_T}} \\
&+ k^{-2\sigma_T} \sum_{q=1}^k |a_q|^2 \sum_{0 \leq m_1 \leq M_q(T)} \sum_{0 \leq m_2 \leq M_q(T)} \frac{1}{(m_1+q/k)^{\sigma_T} (m_2+q/k)^{\sigma_T}} \\
&\times \int_{T_1}^T \left( \frac{m_1+q/k}{m_2+q/k} \right)^{it} dt \\
&+ k^{-2\sigma_T} \sum_{q=1}^k a_{q_1} \sum_{\substack{q_2=1 \\ q_1 \neq q_2}}^k \bar{a}_{q_2} \sum_{0 \leq m_1 \leq M_{q_1}(T)} \sum_{0 \leq m_2 \leq M_{q_2}(T)} \frac{1}{(m_1+q_1/k)^{\sigma_T} (m_2+q_2/k)^{\sigma_T}} \\
&\times \int_{T_1}^T \left( \frac{m_2+q_2/k}{m_1+q_1/k} \right)^{it} dt \\
&+ k^{-2\sigma_T} \sum_{q=1}^k a_{q_1} \sum_{\substack{q_2=1 \\ q_1 \neq q_2}}^k \bar{a}_{q_2} \sum_{0 < m \leq \max(M_{q_1}(T), M_{q_2}(T))} \frac{1}{(m+q_1/k)^{\sigma_T} (m+q_2/k)^{\sigma_T}} \\
&\times \int_{T_1}^T \left( \frac{m+q_2/k}{m+q_1/k} \right)^{it} dt \stackrel{\text{def}}{=} \sum_{j=1}^4 I_{1j}. \tag{2}
\end{aligned}$$

Using the equality [4]

$$\begin{aligned}
\sum_{0 \leq m \leq x} \frac{1}{(m+\alpha)^s} &= \frac{1}{\alpha^s} + \frac{[x]}{(x+\alpha)^s} - s \int_1^x \frac{u-[u]}{(u+\alpha)^{s+1}} du + \frac{s}{s-1} (1+\alpha)^{1-s} \\
&- \frac{s}{s-1} (x+\alpha)^{1-s} + \alpha(1+\alpha)^{-s} - \alpha(x+\alpha)^{-s}, \tag{3}
\end{aligned}$$

where  $0 < \alpha \leq 1$  and  $s \neq 1$ , we find without difficult that

$$I_{11} = \frac{Tl}{4k^{2\sigma_T}}K + \frac{T}{2} \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} + Bk^{-2\sigma_T}K + Bk^{-2\sigma_T}Tl_TK \exp \left\{ -\frac{\log T}{l_T} \right\}. \quad (4)$$

From the estimate [1]

$$\sum_{0 \leq m < n \leq N} \frac{1}{(m + \alpha_1)^\beta (n + \alpha_2)^\theta} \left( \log \frac{n + \alpha_1}{m + \alpha_2} \right)^{-1} = B_c N^{2-\beta-\theta} \log N, \quad (5)$$

where  $0 < \alpha_2, \alpha_1 \leq 1$ ,  $\beta < 1$ ,  $\beta + \theta \leq c < 2$ , we derive that

$$I_{12} = Bk^{-2\sigma_T}KT^{1-\sigma_T} \log T. \quad (6)$$

Similarly

$$I_{13} = Bk^{-2\sigma_T}KT^{1-\sigma_T} \log T,$$

$$I_{14} = Bk^{1-2\sigma_T}KT^{1-\sigma_T} \log T.$$

This and (2), (4) and (6) yield

$$I_1 = \frac{Tl_T}{4}k^{-2\sigma_T}K + \frac{T}{2} \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} + Bk^{-\sigma_T}Tl_TK \exp \left\{ -\frac{\log T}{l_T} \right\} + Bk^{1-2\sigma_T}KT^{1-\sigma_T} \log T. \quad (7)$$

Now let  $T_2 = \max(T/2, 2\pi m_1^2, 2\pi m_2^2)$ . Then

$$\begin{aligned} I_2 &= k^{-2\sigma_T} \sum_{q=1}^k |a_q|^2 \sum_{1 \leq m \leq M_0(T)} \frac{1}{m^{2-2\sigma_T}} \int_{T_2}^T \left( \frac{2\pi}{t} \right)^{2\sigma_T-1} dt \\ &\quad + k^{-2\sigma_T} \sum_{q=1}^k |a_q|^2 \sum_{\substack{1 \leq m_1, m_2 \leq M_0(T) \\ m_1 \neq m_2}} \frac{e^{-2\pi i \frac{q}{k}(m_1-m_2)}}{(m_1 m_2)^{1-\sigma_T}} \\ &\quad \times \int_{T_2}^T \left( \frac{2\pi}{t} \right)^{2\sigma_T-1} \exp \left\{ it \log \frac{m_1}{m_2} \right\} dt \\ &\quad + k^{-2\sigma_T} \sum_{\substack{q_1=1 \\ q_1 \neq q_2}}^k \sum_{q_2=1}^k a_{q_1} \bar{a}_{q_2} \sum_{\substack{1 \leq m_1 \\ m_1 \neq m_2}} \sum_{m_2 \leq M_0(T)} \frac{e^{-\frac{2\pi i}{k}(q_1 m_1 - q_2 m_2)}}{(m_1 m_2)^{1-\sigma_T}} \end{aligned}$$

$$\begin{aligned}
 & \times \int_{T_2}^T \left(\frac{2\pi}{t}\right)^{2\sigma_T-1} \exp\left\{it \log \frac{m_1}{m_2}\right\} dt \\
 & + k^{-2\sigma_T} \sum_{\substack{q_1=1 \\ q_1 \neq q_2}}^k \sum_{q_2=1}^k a_{q_1} \bar{a}_{q_2} \sum_{1 \leq m \leq M_0(T)} \frac{e^{-\frac{2\pi i m}{k}(q_1 - q_2)}}{m^{2-2\sigma_T}} \int_{T_2}^T \left(\frac{2\pi}{t}\right)^{2\sigma_T-1} dt \\
 & \stackrel{\text{def}}{=} \sum_{j=1}^4 I_{2j}.
 \end{aligned} \tag{8}$$

The application of (3) shows that

$$I_{21} = k^{-2\sigma_T} K \frac{T}{4} l_T \left(1 + B \exp\left\{-\frac{\log T}{l_T}\right\} + B l_T^{-1}\right), \tag{9}$$

$$I_{24} = B k^{1-2\sigma_T} K T l_T \exp\left\{-\frac{\log T}{l_T}\right\}. \tag{10}$$

Since

$$\int_{T_2}^T \left(\frac{t}{2\pi}\right)^{1-2\sigma_T} \exp\left\{it \log \frac{m_1}{m_2}\right\} dt = B T^{1-2\sigma_T} \left|\log \frac{m_1}{m_2}\right|^{-1},$$

in view of (5) we obtain

$$I_{22} = B k^{-2\sigma_T} K T^{1-\sigma_T} \log T,$$

$$I_{23} = B k^{1-2\sigma_T} K T^{1-\sigma_T} \log T.$$

Hence and from (8)–(10) we have

$$I_2 = k^{-2\sigma_T} K \frac{T l_T}{4} + B k^{1-2\sigma_T} K T l_T \exp\left\{-\frac{\log T}{l_T}\right\} + B k^{-2\sigma_T} K T. \tag{11}$$

It is easily seen that

$$I_3 = B k^{1-2\sigma_T} K T^{1-\sigma_T}, \tag{12}$$

$$I_4 = B k^{1-2\sigma_T} K. \tag{13}$$

Repeating the proof of Theorem 2 from [3], we find that

$$I_5 = B k^{1-2\sigma_T} K T^{1/2} \log T,$$

$$I_6 = B k^{1-2\sigma_T} K T^{1-\sigma_T} \log T + B k^{1-\sigma_T} K T^{(1-\sigma_T)/2},$$

$$I_7 = B k^{1-2\sigma_T} K T^{1-\sigma_T} \log T,$$

$$I_8 + I_9 + I_{10} = B k^{1-\sigma_T} K T^{1/2}.$$

This and (10), (7), (11)–(13) show that

$$\int_{T/2}^T |\zeta(\sigma_T + it; \mathbf{a})|^2 dt = k^{-2\sigma_T} K \frac{T l_T}{2} + \frac{T}{2} \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} \\ + B k^{1-2\sigma_T} K T l_T \exp \left\{ -\frac{\log T}{l_T} \right\} + B k^{-2\sigma_T} K T + B k^{1-\sigma_T} K T^{1/2} \log T.$$

Taking  $2^{-j}T$  instead of  $T$  and summing over all  $j \geq 0$ , hence we deduce Theorem 1.

*Proof of Theorem 2.* Clearly, it suffices to calculate the integrals  $I_1$  and  $I_2$  in (1). In this case we find that

$$I_1 = k^{-2\sigma_T} K \frac{T \log T}{4} + \frac{T}{2} \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} + B k^{-2\sigma_T} K T \\ + B k^{2-\sigma_T} K \frac{T \log T}{l_T} + B k^{1-2\sigma_T} K T^{1-\sigma_T} \log T, \\ I_2 = k^{-2\sigma_T} K \frac{T \log T}{4} + B k^{1-2\sigma_T} K T + B k^{-2\sigma_T} + K \frac{T \log T}{l_T},$$

and the proof is completed in the same way as Theorem 1.

*Proof of Theorem 3.* In this case we have

$$I_1 = k^{-2\sigma_T} K \frac{\kappa}{4} (1 - e^{-1/\kappa} T \log T + \frac{T}{2} \sum_{q=1}^k \frac{|a_q|^2}{q^{2\sigma_T}} \\ + o(k^{-2\sigma_T} K T \log T) + B k^{1-2\sigma_T} K T^{1-\sigma_T} \log T, \\ I_2 = k^{-2\sigma_T} K \frac{\kappa}{4} (e^{-1/\kappa} - e^{-2/\kappa}) T \log T + o(k^{-2\sigma_T} K T \log T) + B k^{1-2\sigma_T} K T.$$

## References

- [1] R. Garunkštis, A. Laurinčikas, J. Steuding, On the mean square of Lerch zeta-function, *Arch. Math.* (to appear).
- [2] A. Kačėnas, A. Laurinčikas, On the periodic zeta-function, *Lith. Math. J.*, **41** (2), 168–177 (2001).
- [3] A. Laurinčikas, D. Šiaučiūnas, On the periodic zeta-function. II, *Lith. Math. J.*, **41** (4), 361–372 (2001).
- [4] A. Laurinčikas, J. Steuding, The mean square of the Lerch zeta-function near the critical line, *Liet. matem. rink.*, **43** (2), 213–226 (2003).
- [5] D. Šiaučiūnas, On the mean square for the periodic zeta-function on the critical line, *Liet. matem. rink.*, **41** spec. nr., 128–133 (2001).

## Apie periodinės dzeta funkcijos kvadrato vidurki

A. Kačėnas, D. Šiaučiūnas

Gauta periodinės dzeta funkcijos antrojo momento asimptotika arti kritinės tiesės.