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An approximation of the solution of Stratanovich integral equation driven by a continuous p-semimartingale

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Introduction

Consider the Stratonovich integral equation

$$X_t = \eta + (S)\int_0^t f(X_s)\,\mathrm{d}Z_s, \quad t \in [0, T], \tag{1}$$

or equivalent equation

$$X_t = \eta + \int_0^t f(X_s)\,\mathrm{d}Z_s + \frac{1}{2}\int_0^t ff'(X_s)\,\mathrm{d}s, \quad t \in [0, T],$$

where $Z = W + B^H$, W is a standard Brownian motion, B^H is a fractional Brownian motion (fBm) with Hurst index $1/2 < H < 1$. For short, we shall write $ff'(X_s)$ instead of $f(X_s)f'(X_s)$.

In compute simulation of the solution of the equation (1) it is useful to find a good approximation. The main problem is to approximate fBm B^H. Several schemes of an approximations of the fBm are considered in [1,3–5].

Assume that the self similarity index H satisfies $H > 1/2$. In this case we have the following kernel representation of B^H with respect to the standard Brownian motion

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s$$

with a deterministic kernel

$$K_H(t,s) = c_H\Big(H - \frac{1}{2}\Big)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\,\mathrm{d}u,$$

where c_H is the normalizing constant.

Let $\varkappa^n = \{t_k^n\colon 0 \leqslant k \leqslant n\}$, $n \geqslant 1$, be a sequence of partitions of the interval $[0, T]$, i.e., $0 = t_0^n < t_1^n < \cdots < t_n^n = T$, $t_k^n = Tk/n$.

For partition $\varkappa^n$ define $\rho^n(t) = \max\{t_k^n\colon\ t_k^n \leqslant t\}$ and $r^n(t) = \max\{k\colon\ t_k^n \leqslant t\}$, $t \in [0,T]$. For every $x \in D([0,T])$ the sequence $(x^{\varkappa^n})$ denotes the following discretizations of x:

$$x_t^{\varkappa^n} = x(t_k^n) \ \text{ for } t \in [t_k^n, t_{k+1}^n), \quad 0 \leqslant k \leqslant n, \ n \in \mathbb{N}.$$

Define

$$A_t^n = \sum_{k=1}^{r^n(t)} K_H(\rho^n(t), t_k^n)\big(W(t_k^n) - W(t_{k-1}^n)\big), \quad M_t^n = W_t^{\varkappa^n}.$$

Let $\widehat{Z}^n$ and $\widetilde{Z}^n$ be linear approximations of the processes $Z^n = M^n + A^n$ and $Z = W + B^H$ correspondingly, i.e.,

$$\widehat{Z}_t^n = Z^n(t_{k-1}^n) + \frac{t - t_{k-1}^n}{t_k^n - t_{k-1}^n}\big(Z^n(t_k^n) - Z^n(t_{k-1}^n)\big) \ \text{ for } t \in [t_{k-1}^n, t_k^n),$$

$$\widetilde{Z}_t^n = Z(t_{k-1}^n) + \frac{t - t_{k-1}^n}{t_k^n - t_{k-1}^n}\big(Z(t_k^n) - Z(t_{k-1}^n)\big) \ \text{ for } t \in [t_{k-1}^n, t_k^n),$$

where $n \in \mathbb{N}$, $1 \leqslant k \leqslant n$. Define the approximations

$$Y_t^n = \eta + \int_0^t f(Y_s^n)\,\mathrm{d}\widehat{Z}_s^n, \quad \widetilde{Y}_t^n = \eta + \int_0^t f(\widetilde{Y}_s^n)\,\mathrm{d}\widetilde{Z}_s^n, \quad t \in [0,T].$$

Now we formulate our results.

THEOREM 1. *Let f' be a continuous function and $f(x) > 0$ for all x. Then* $\sup_{t \leqslant T} |Y_t^n - X_t| \xrightarrow{\mathbf{P}} 0$ *as* $n \to \infty$.

THEOREM 2. *Let $f \in C_b^3(\mathbb{R})$. Then*

$$n^{1/q}(1 + \ln n)^{-1/2} \sup_{t \leqslant T} \big|\widetilde{Y}_t^n - X_t\big| \xrightarrow{\mathbf{P}} 0 \ \text{ as } \ n \to \infty.$$

1. Proofs

LEMMA 1. *We have* $\sup_{t \leqslant T} |A_t^n - B_t^H| \xrightarrow{\mathbf{P}} 0$ *as* $n \to \infty$.

Proof. For every fix $t > 0$ and $\varepsilon > 0$

$$\mathbf{E}\big|A_t^n - B_t^H\big|^2$$
$$\leqslant 4\mathbf{E}\bigg|\sum_{k=1}^{r^n(\varepsilon)} K_H\big(\rho^n(t), t_k^n\big)\big(W(t_k^n) - W(t_{k-1}^n)\big)\bigg|^2 + 4\mathbf{E}\bigg|\int_0^{\rho^n(\varepsilon)} K_H(t,s)\,\mathrm{d}W_s\bigg|^2$$

$$+4\mathbf{E}\Big|\sum_{k=r^n(\varepsilon)+1}^{r^n(t)}\Big(K_H\big(\rho^n(t),t_k^n\big)\big(W(t_k^n)-W(t_{k-1}^n)\big)-\int_{t_{k-1}^n}^{t_k^n}K_H(t,s)\,\mathrm{d}W_s\Big)\Big|^2$$

$$+4\mathbf{E}\Big|\int_{\rho^n(t)}^{t}K_H(t,s)\,\mathrm{d}W_s\Big|^2.$$

By martingale property and inequality

$$|K_H(t,s)|\leqslant c_H(H-1/2)^{-1}s^{1/2-H}:=\widehat{c}_H\,s^{1/2-H}$$

we get

$$\mathbf{E}\big|A^n-B_t^H\big|^2\leqslant 4n^{-1}\sum_{k=1}^{r^n(\varepsilon)}K_H^2\big(\rho^n(t),t_k^n\big)+4\int_0^{\rho^n(\varepsilon)}K_H^2(t,s)\,\mathrm{d}s$$

$$+4\sum_{k=r^n(\varepsilon)+1}^{r^n(t)}\int_{t_{k-1}^n}^{t_k^n}\big(K_H\big(\rho^n(t),t_k^n\big)-K_H(t,s)\big)^2\mathrm{d}s+4\int_{\rho^n(t)}^{t}K_H^2(t,s)\,\mathrm{d}s$$

$$\leqslant 8\widehat{c}_H^2\,\frac{\varepsilon^{2-2H}}{2-2H}+\frac{4c_H^2}{(H-1/2)^2n^{2H-1}}\Big\{3\Big(\frac{1}{\rho^n(\varepsilon)}\Big)^{H-1/2}+1\Big\}^2+\frac{4\widehat{c}_H^2}{2-2H}\,\frac{1}{n^{2-2H}}.$$

Thus $\mathbf{E}|A_t^n-B_t^H|^2\to 0$ as $n\to\infty$ for every $t>0$.

By simple calculations we get

$$\mathbf{E}\big|A^n(t)-A^n(s)\big|^2\leqslant\frac{2c_H}{(H-1/2)^2}\,\frac{1}{2-2H}\,\big(\rho^n(t)-\rho^n(s)\big)^{2H-1}.$$

By tightness criterium formulated in Theorem 6.4.1 [2]we get that the sequence (A^n) is tight. The sequence A^n-B^H is tight as a difference of two tight sequences (see Corollary 3.3.3 in section 6 [2]). Thus we obtain the statement of the lemma.

Proof of Theorem 1. From [6] one can get that

$$\big|Y_t^n-X_t\big|\leqslant\Big(1+\sup_{t\leqslant T}|X_t|\Big)\Big(\exp\big\{C_1|\widehat{Z}_t^n-Z_t|\big\}-1\Big),$$

where C_1 is a constant. Since

$$\sup_{t\leqslant T}|\widehat{Z}_t^n-\widehat{Z}_t^{n,\varkappa^n}|\leqslant\max_{1\leqslant k\leqslant n}\big|Z^n(t_k^n)-Z^n(t_{k-1}^n)\big|$$

$$\leqslant\max_{1\leqslant k\leqslant n}\big|W(t_k^n)-W(t_{k-1}^n)\big|+\max_{1\leqslant k\leqslant n}\big|A^n(t_k^n)-A^n(t_{k-1}^n)\big|$$

and

$$\mathbf{E}\max_{1\leqslant k\leqslant n}\big|A^n(t_k^n)-A^n(t_{k-1}^n)\big|\leqslant C_2n^{1/2-H}(1+\ln n)^{1/2},$$

then $\sup_{t\leqslant T}|\widehat{Z}_t^n - Z_t| \xrightarrow{\mathbf{P}} 0$ as $n\to\infty$. Thus we proof is completed.

Proof of Theorem 2. Let

$$X_t^n = \eta + \int_0^t f(X_s^{n,\varkappa^n})\,\mathrm{d}Z_s + \frac{1}{2}\int_0^t ff'(X_s^{n,\varkappa^n})\,\mathrm{d}s + \int_0^t ff'(X_s^{n,\varkappa^n})\biggl(\int_{\rho_s^n}^s \mathrm{d}W_u + \int_{\rho_s^n}^s \mathrm{d}B_u^H\biggr)\mathrm{d}B_s^H.$$

Similarly as in [3] one can show that

$$n^{1/q}(1+\ln n)^{-1/2}V_q\bigl(X - X^n; [0,T]\bigr) \xrightarrow{\mathbf{P}} 0 \quad n\to\infty, \quad \text{if } q>2.$$

It is easy to show that

$$\mathbf{E}\sup_{t\leqslant T}|X_t^n - X_t^{n,\varkappa^n}| \leqslant C_3 n^{-1/2}(1+\ln n)^{1/2},$$

$$\mathbf{E}\sup_{t\leqslant T}|Y_t^n - Y_t^{n,\varkappa^n}| \leqslant C_4\bigl\{n^{-H} + n^{-1/2}\bigr\}(1+\ln n)^{1/2}.$$

It still remains to prove that

$$n^{1/2}(1+\ln n)^{1/2}\sup_{t\leqslant T}|X_t^{n,\varkappa^n} - Y_t^{n,\varkappa^n}| \xrightarrow{\mathbf{P}} 0 \ \text{ as } \ n\to\infty.$$

The proof of this fact is too long for this note.

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REZIUMĖ

K. Kubilius. Stratanovičiaus integralinės lygties sprendinio, valdomo tolydaus p-semimartingalo, aproksimacija

Konstruojamos dvi Vong-Zakai tipo aproksimacijos. Gautos sąlygos, kada pirmoji aproksimacija konverguoja pagal tikimybę į Stratanovičiaus integralinės lygties sprendinį. Rastas antrosios aproksimacijos konvergavimo greitis.