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Estimation of parameters in ageing model using Bayesian approach

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1. Introduction

Commonly Bayesian approach is applied to update estimated parameters of stationary process when more statistical information becomes available. Because of failure rate increasing, in the ageing models the processes become non-stationary. In such case available statistical data can not be used for updating failure rate of previous time period, because it represents the other state of equipment.

Sometimes approximate failure rate dependence on time for the particular groups of equipments is priory known. The parameters of this dependence are stationary variables. However prior information can lead some uncertainty and they are assumed as random variables with prior probability distributions. These distributions (and also variation expression of failure rate) are updated by Bayesian approach using available statistical data.

The authors did not succeed to find any examples of that method in literature. Therefore this paper presents the main idea of developed algorithm and numerical example.

In order to estimate time dependent random parameters the iterative process based on Bayesian approach was used. Reliability estimates when new statistical data become available could be updated by using Bayes theorem:

$$p(\theta \mid E) = \frac{p(\theta)L(E \mid \theta)}{\int p(u)L(E \mid u)\,\mathrm{d}u}, \tag{1}$$

where $p(\theta \mid E)$ – the posterior distribution of parameter θ in light of statistical data E; $p(\theta)$ - the prior distribution of parameter θ; $L(E \mid \theta)$ – the likelihood that statistical data E would be observed if value of random parameter was θ.

In general, applying Bayesian approach various formal techniques are used in order to estimate posterior distribution, however most of them are too complicated for practical computations. Depending on likelihood function, some prior distributions can always lead to posterior distribution, which has the same functional form as the prior distribution and this functional can be computed analytically, i.e., without introducing numerical error. This useful statistical property is related with so-called conjugate pair of prior distribution and likelihood. Using the conjugate pairs the mean and variance as well as other parameters can be easily estimated for posterior distribution in case prior distribution parameters are available. Thus such conjugate pairs are also proposed to be used in considered iterative estimation of reliability parameters [2,3].

2. Iterative estimation

Usually statistical data of modelled object became available partly and Bayesian approach has to be applied iteratively in order to update prior distribution. In this case, analytical expression of posterior distribution is very useful. Considering reliability information the Beta, Gamma, Normal, Lognormal and others distributions are used as reliability parameters distributions. It is easy to prove that some of them are conjugated and can be updated iteratively as shown in the following examples.

Example 1. Lets assume that the statistical data is given, i.e., $y_i = (k_i, t_i)$, $i = 1, \ldots, n$, where k_i is the number of failures during period t_i. Besides, the number of failures has Poisson distribution with unknown parameter λ. The prior distribution of unknown parameter λ is modelled as Gamma distribution with the density function denoted as

$$p(x) = \frac{a^b}{\Gamma(b)} x^{b-1} \mathrm{e}^{-ax} = g(x, a, b), \quad a > 0, \quad b > 0. \tag{2}$$

Using Bayes' approach prior density function of random parameter λ after j iterations became posterior:

$$p(x \mid y_i, \ldots, y_j) = \frac{p(x \mid y_i, \ldots, y_{j-1}) L(y_j \mid x)}{\int_0^\infty p(u \mid y_i, \ldots, y_{j-1}) L(y_j \mid u)\, \mathrm{d}u}; \tag{3}$$

i.e.,

$$p(x \mid y_i, \ldots, y_j) = g(x, a + t_1 + \ldots + t_j, b + k_1 + \ldots + k_j). \tag{4}$$

Posterior distribution, which is obtained using Bayes' approach belongs to the same parametric distributions family as prior. This means that Poisson likelihood and Gamma distribution are conjugate pair.

By analogical calculation another four conjugated pairs of probabilistic distributions were obtained:

$Ga(a, b) - Ga(a_0, b_0)$ (parameter a of Gamma distribution is modelled by prior Gamma distribution);

$B(p) - Be(a, b)$ (parameter p of Binomial distribution is modelled by prior Beta distribution);

$N(\mu, \sigma) - N(\mu_0, \sigma_0)$ (parameter μ of Normal distribution is modelled by prior Normal distribution);

$Logn(\mu, \sigma) - N(\mu_0, \sigma_0)$ (parameter μ of Lognormal distribution is modelled by prior Normal distribution).

Usually before the estimation of random parameters some statistical data y_i, $i = 1, \ldots, n$ (n – number of observations) are already collected. If data are obtained periodically we can estimate iteratively in order to update posterior distribution by all available data. It was observed, that new information can be joined into the prior distribution in two ways: using Bayes' approach iteratively or in one final step. The result is the same in both ways.

Example 2. Lets assume that the distribution of statistical data $y_i = (k_i, t_i)$, $i = 1, \ldots, n$ (where k_i is the number of failures during period t_i) is Poisson with unknown parameter λ. Define $T_j = t_1 + \ldots + t_j$ and $K_j = k_1 + \ldots + k_j$, $j = 1, \ldots, n$. In that case, probabilistic distribution of K_j is Poisson with parameter λT_j. Random parameter λ is modelled by Gamma distribution like in example 1. After one iteration using Bayesian approach we have posterior λ distribution with density function:

$$p(x \mid K_j, T_j) = g(x, a + T_j, b + K_j). \tag{5}$$

This is the same posterior distribution as in example 1. So the data integration result (posterior distribution) is the same regardless of whether it has been obtained iteratively (by applying Bayes' approach in steps) or in one step (by applying Bayes' approach to cumulative information). It should be noted that all probabilistic distributions (not only conjugated) which are obtained using Bayesian approach have the same property, if the assumption that considered parameter (λ) don't depend on time, is used. In this case the data integration in one step is very useful for prior distribution practical development.

3. Time dependent failure rate

In this section the case when random parameter represents non-stationary process is considered. Using such parameter there is possibility to model the ageing process. Usually ageing process is described by failure rate defined as time function $\lambda(t)$. Let's assume that failure rate function has one unknown parameter with some prior distribution. Then prior distribution can be updated by statistical data applying Bayes approach. However, it should be noted that $\lambda(t, a)$ is used instead of $\lambda(t)$ in likelihood function and Bayesian approach is applying with regard of unknown parameter a. In that case our result is posterior distribution of parameter a. Let's assume the failure number k_i, $i = 1, 2, \ldots, n$, follows Poisson distribution with time dependent failure rate $\lambda(i) = ai$, $a > 0$, where unknown parameter a has probability distribution with density function $p(x)$. Using Bayes' approach posterior distribution of parameter a is

$$p(x|k_1, \ldots, k_j) = \frac{p(x) \cdot x^{K_j} \mathrm{e}^{-I_j x}}{\int_0^\infty p(u) \cdot u^{K_j} \mathrm{e}^{-I_j u}\, \mathrm{d}u}, \tag{6}$$

where $I_j = \frac{j(j+1)}{2}$, $K_j = \sum\limits_{i=1}^{j} k_i$, $j = 1, \ldots, n$.

Usually the empirical mean of failure rate is calculated only in engineering reliability estimations. When failure rate depends on time, such estimations can lead high level of errors. Especially it is important when rare events are considered and available failure statistics are not enough. This situation is typical in reliability estimation for nuclear energetic objects. In this case, the prior information of failure rate or the distributions of others parameters are very important. Generic statistic data of similar nuclear object failure are collected in T-book [5], and such information can be used in estimations as prior by applying Bayesian approach.

In this section numerical example of modeling is presented. Let's assume, that failure number k_i, $i = 1, 2, \ldots, 50$, in the i-th interval of time is simulated by Poison distribution with parameter $\lambda^*(i) = ai$. For simulation k_i the value of parameter a is chosen $a = 0.05$. Prior distribution of unknown parameter a is assumed as Gamma with parameters $\mu = 2$ and $\eta = 0.5$ in the mathematical model. Then density function of parameter a distribution is

$$p(x) = \frac{2^{0.5}}{\Gamma(0.5)} x^{0.5-1} e^{-2x} = g(x, 2, 0.5). \tag{7}$$

Posterior distribution of parameter a is estimated using simulated data of failure number $k_1, k_2, \ldots, k_{50}$ in time intervals $[0; 1], (1; 2], \ldots, (49; 50]$ and formula (6). So density function of posterior distribution is

$$p(x|k_1, \ldots, k_i) = \frac{(2 + I_i)^{0.5+K_i}}{\Gamma(0.5 + K_i)} x^{0.5+K_i-1} e^{-x(2+I_i)} = g\left(x, 2 + I_i, 0.5 + K_i\right), \tag{8}$$

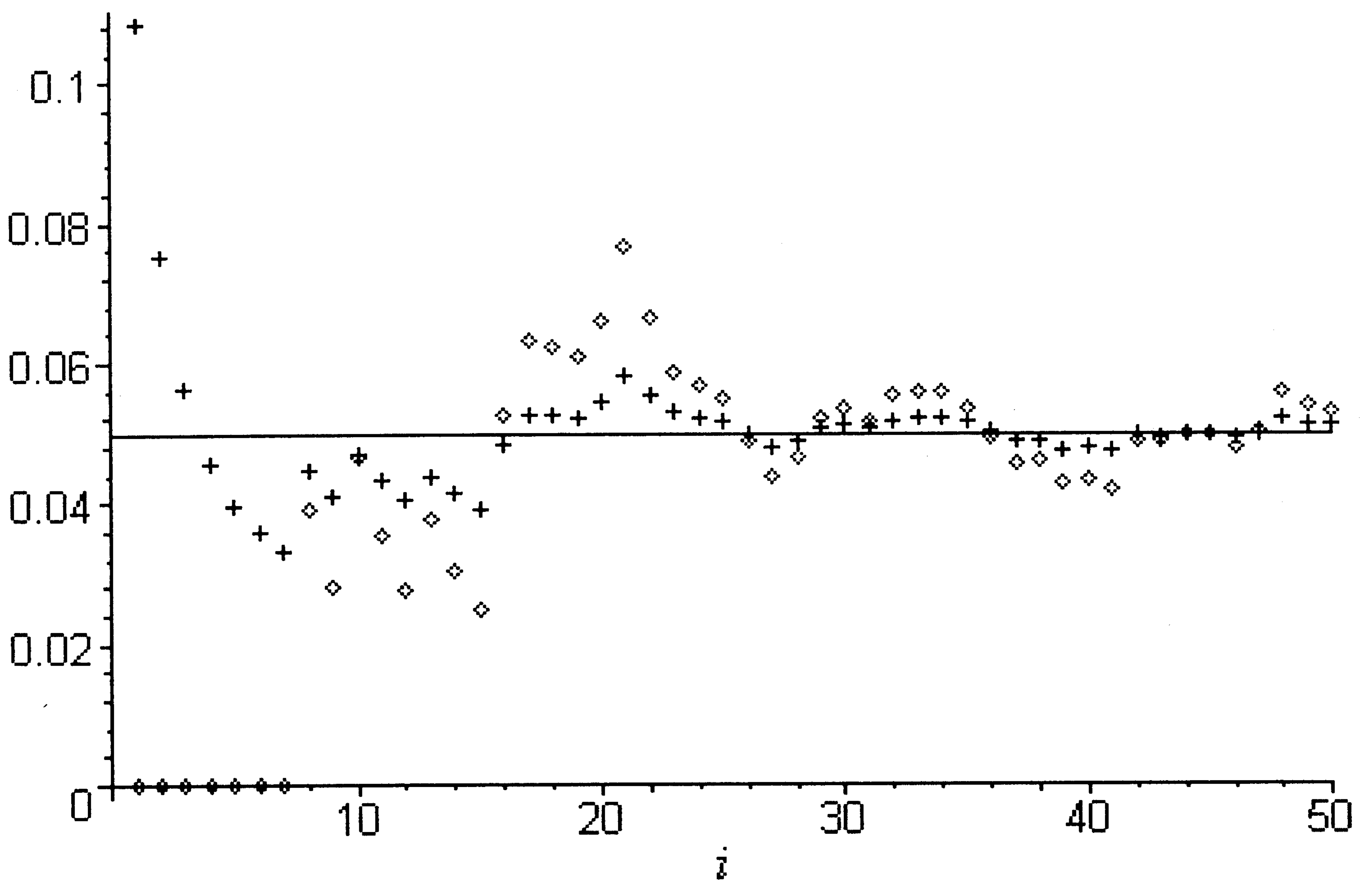

Fig. 1. Calculated estimations of parameter a. + – calculated estimations of parameter a using Bayesian approach; ◇ – calculated estimations of parameter a using least squares method; – – true value of parameter a (i.e., $a = 0.05$).

and the mean of parameter a is

$$\hat{a}_i = E(x|k_1, \ldots, k_i) = \frac{0.5 + K_i}{2 + I_i}, \quad i = 1, 2, \ldots, 50. \tag{9}$$

Usually least squares method is used to valuate parameter a in engineering estimations, when the dependence function is known beforehand. The obtained estimations of parameter a using Bayesian approach and least squares method are presented in Fig. 1.

The results of considered numerical example show that calculated unknown parameter estimate using Bayesian approach is more precise then the other one calculated using least squares method in this case.

4. Conclusions

The pairs of probabilistic distribution: Poisson – Gamma, Gamma – Gamma, Binomial – Beta, Normal – Normal and Lognormal – Normal are conjugated and can be applied in estimation of reliability parameters without numerical method error. Bayes approach allows update initial reliability parameters iteratively or in one final step.

The method how to estimate parameters related to an ageing process is presented in this paper. Using numerical example it was shown iterative estimation of reliability parameter applying Bayesian approach. And calculated results were compared with the other ones calculated using least squares method.

References

1. G.E.P. Box, G.C. Tio, *Bayesian Inference in Statistical Analysis*, Wiley-Interscience (1992).
2. S.A. Aivazjan, V.S. Mchitarjan, *Prikladnaja statistika v zadach i upraznienijah*, Juniti-Dana, Moskva (2001).
3. J.M.Bernardo, A.F.M. Smith, *Bayesian Theory*, John Wiley and Sons (2003).
4. C.A. Clarotti, A. Lannoy, H. Procaccia, A Bayesian reliability growth model for ageing components, in: *Safety and Reliability* (1998).
5. *T-book, Reliability Data of Components in the Nordic Nuclear Power Plants*, third edition, prepared by The ATV office and Studsvik AB, Stockholm, Sweden (1992).

REZIUMĖ

J. Augutis, I. Žutautaitė, E. Augutienė. Senėjimo modelio parametrų vertinimas naudojant Bajeso metodą

Šiame straipsnyje nagrinėjamas jungtinių tikimybinių skirstinių porų taikymas patikimumo parametrų įvertinimui Bajeso rekurentinėje formulėje. Sudarytas algoritmas senėjimo modelio parametrų įvertinimui, panaudojant Bajeso formulę, ir pateiktas šio algoritmo taikymo skaitinis pavyzdys.