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The distributions of sums of the prime indicators with respect to distinct frequencies

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Abstract. A characterization of a set of strongly additive functions f_x which has the same limit law with respect to ordinary and logarithmic frequencies is obtained. Strongly additive functions which take zero or unit for each prime p are considered.

Keywords: additive function, logarithmic frequency, limit distribution.

Introduction

In the present paper we consider the distribution of integer valued strongly additive function f_x, $x \geqslant 2$. The additive function f_x is allowed to depend on x in arbitrary way. But we investigate only those strongly additive functions for which $f_x(p) \in \{0, 1\}$ over primes p. Here are some examples of such functions:

$$f_x^*(n) = \sum_{\substack{p|n \\ p<\ln x}} 1, \quad \hat{f}_x(n) = \sum_{\substack{p|n \\ \ln x<p\leqslant \ln^2 x}} 1, \quad \tilde{f}_x(n) = \sum_{\substack{p|n \\ \sqrt{x}<p\leqslant x}} 1.$$

The distribution of the additive function f_x we consider with respect to the frequencies

$$\nu_x(A) = \frac{1}{[x]} \sum_{\substack{n\leqslant x \\ n\in A}} 1, \quad \mu_x(A) = \Big(\sum_{n\leqslant x} \frac{1}{n}\Big)^{-1} \sum_{\substack{n\leqslant x \\ n\in A}} \frac{1}{n}.$$

In Levin-Timofeev paper [1] it was shown that the conditions

$$\nu_x\Big(\frac{f(n) - \alpha(x)}{\beta(x)} < u\Big) \underset{x\to\infty}{\Rightarrow} F(u), \quad \mu_x\Big(\frac{f(n) - \alpha(x)}{\beta(x)} < u\Big) \underset{x\to\infty}{\Rightarrow} \Phi(u),$$

where $f(n)$ is an additive function, $\alpha(x)$ is a real function, $\beta(x)$ is the positive unbounded increasing function and $\Phi(u)$ is proper distribution function, imply

$$\frac{1}{\ln x} \sum_{\substack{p\leqslant x \\ f(p)\leqslant u\beta(x)}} \frac{\ln p}{p} \underset{x\to\infty}{\Rightarrow} L(u)$$

for some distribution function $L(u)$. From the main result of this paper we can see that the weak convergence conditions

$$\nu_x(f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u) \qquad \mu_x(f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u)$$

yield similar relation:

$$\lim_{x\to\infty} \frac{1}{\ln x} \sum_{\substack{p\leqslant x\\ f(p)=1}} \frac{\ln p}{p} = 0.$$

The aim of this work is to prove the following assertion.

THEOREM. *Let $\{f_x, x \geqslant 2\}$, be a set of strongly additive functions and $f_x(p) \in \{0, 1\}$ for each prime number p. The next three assertions are equivalent:*

(A) *The distributions $\nu_x(f_x(n) < u)$ and $\mu_x(f_x(n) < u)$ converge weakly to the same limit law.*

(B) *The limits*

$$\lim_{x\to\infty} \sum_{p_1\leqslant x}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}}^{*} \frac{1}{p_1p_2\cdots p_l} = g_l \tag{1}$$

exist for each fixed positive integer l and

$$\lim_{x\to\infty} \frac{1}{\ln x} \sum_{p\leqslant x}^{*} \frac{\ln p}{p} = 0, \tag{2}$$

*where the superscript * over the sign of the sum means that the summation is expanded over primes for which $f_x(p) = 1$.*

(C) *The condition* (2) *is satisfied and*

$$P\Big(\sum_{p\leqslant x} \xi_{xp} < u\Big)$$

converges weakly. Here ξ_{xp}, $p \leqslant x$, are independent random variables for fixed x, and for prime number $p \leqslant x$:

$$P(\xi_{xp} = 1) = \begin{cases} \frac{1}{p} & \textit{if} \quad f_x(p) = 1, \\ 0 & \textit{if} \quad f_x(p) = 0, \end{cases} \qquad P(\xi_{xp} = 0) = \begin{cases} 1 - \frac{1}{p} & \textit{if} \quad f_x(p) = 1, \\ 1 & \textit{if} \quad f_x(p) = 0. \end{cases}$$

Preliminary results

The main result follows from next three lemmas.

LEMMA 1. *Let $f_x, x \geqslant 2$, be a set of strongly additive functions from the theorem. The distribution functions $\nu_x(f_x(n) < u)$ converge weakly as $x \to \infty$ if and only if the* *condition* (1) *is satisfied. Moreover, the limit law has the characteristic function*

$$1+\sum_{l=1}^{\infty}\frac{g_l}{l!}(e^{it}-1)^l.$$

Proof of this lemma is given in [2].

LEMMA 2. *Let $f_x, x \geqslant 2$, be a set of strongly additive functions from the theorem. The distribution functions $\mu_x(f_x(n)<u)$ converge weakly as $x\to\infty$ if and only if the limits*

$$\lim_{x\to\infty}\sum_{p_1\leqslant x}^{*}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*}\cdots\sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}}^{*}\frac{1}{p_1p_2\dots p_l}\left(1-\frac{\ln p_1p_2\dots p_l}{\ln x}\right)=h_l \tag{3}$$

exist for any positive integer l. Moreover, in this case the limit distribution has the characteristic function

$$1+\sum_{l=1}^{\infty}\frac{h_l}{l!}(e^{it}-1)^l.$$

The proof of this lemma can be found in [3].

LEMMA 3. *Let $f_x, x\geqslant 2$, be a set of real-valued strongly additive arithmetic functions. Let η_{xp} be independent random variables, defined for each prime p in the range $2\leqslant p\leqslant x$, and distributed according to*

$$\eta_{xp}=\begin{cases} f_x(p) & \text{with probability } \frac{1}{p},\\ 0 & \text{with probability } 1-\frac{1}{p}.\end{cases}$$

Then there is a positive absolute constant such that the inequality

$$\rho\Big(\nu_x\big(f_x(n)<u\big),P\Big(\sum_{p\leqslant x}\eta_{xp}<u\Big)\Big)$$
$$\ll\Big(\sum_{\substack{x^{\varepsilon}<p\leqslant x\\ |f_x(p)|>v}}\frac{1}{p}+\frac{v}{\varepsilon}+\exp\Big\{-\frac{1}{8\varepsilon}\ln\frac{1}{\varepsilon}\Big\}+x^{-1/15}\Big)$$

holds uniformly for all $v>0, 0<\varepsilon<1, x\geqslant 2$. Here $\rho(F(u),G(u))$ denotes the Levy distance between distribution functions $F(u)$ and $G(u)$.

This last lemma is proved in [4].

The sketch of the proof

I. (A)⇒(B). Let $F(u)$ be the distribution function such that

$$\nu_x(f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u), \qquad \mu_x(f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u).$$

It follows from Lemmas 1 and 2 it follows that the conditions (1) and (3) are satisfied for every fixed positive integer l. Since $F(u)$ is the common limit distribution function for $\nu_x(f_x(n) < u)$ and $\mu_x(f_x(n) < u)$, we have that $g_l = h_l$ for every positive integer l. In particular $g_1 = h_1$. Hence

$$\lim_{x\to\infty} \frac{1}{\ln x} \sum_{p\leqslant x}^{*} \frac{\ln p}{p} = \lim_{x\to\infty} \left(\sum_{p\leqslant x}^{*} \frac{1}{p} - \sum_{p\leqslant x}^{*} \frac{1}{p}\left(1 - \frac{\ln p}{\ln x}\right)\right) = g_1 - h_1 = 0.$$

Thus, the condition (2) is satisfied.

II. (B)⇒(C). The condition (1) implies that

$$\nu_x(f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u)$$

for some distribution function $F(u)$. From Lemma 3 and condition (2) it follows that

$$\lim_{x\to\infty} \rho\left(\nu_x(f_x(n) < u), P\left(\sum_{p\leqslant x} \xi_{xp} < u\right)\right) = 0. \tag{4}$$

Therefore

$$P\left(\sum_{p\leqslant x} \xi_{xp} < u\right) \underset{x\to\infty}{\Rightarrow} F(u).$$

III. (C)⇒(A). Let $F(u)$ is the limit distribution function for the sum

$$\sum_{p\leqslant x} \xi_{xp}.$$

Using the condition (2) and Lemma 3 we obtain the relation (4). Hence

$$\nu_x(f_x(n) < u) \underset{x\to\infty}{\Rightarrow} F(u).$$

Lemma 1 implies (1). Having in mind that

$$\lim_{x\to\infty} \sum_{p\leqslant x}^{*} \frac{1}{p} = g_1$$

and

$$\sum_{p_1\leqslant x}^{*} \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}^{*} \cdots \sum_{\substack{p_l\leqslant x\\ p_l\neq p_1,p_2,\dots,p_{l-1}\\ p_1p_2\cdots p_l\leqslant x}}^{*} \frac{\ln(p_1p_2\dots p_l)}{p_1p_2\dots p_l \ln x}$$

$$\leqslant \sum_{j=1}^{l} \frac{1}{\ln x} \sum_{p_1 \leqslant x}^{*} \sum_{p_2 \leqslant x}^{*} \cdots \sum_{p_l \leqslant x}^{*} \frac{\ln p_j}{p_1 p_2 \ldots p_l} = \frac{l}{\ln x} \sum_{p \leqslant x}^{*} \frac{\ln p}{p} \left(\sum_{p \leqslant x}^{*} \frac{1}{p} \right)^{l-1}$$

we obtain

$$\lim_{x \to \infty} \sum_{p_1 \leqslant x}^{*} \sum_{\substack{p_2 \leqslant x \\ p_2 \neq p_1}}^{*} \cdots \sum_{\substack{p_l \leqslant x \\ p_l \neq p_1, p_2, \ldots, p_{l-1} \\ p_1 p_2 \cdots p_l \leqslant x}}^{*} \frac{1}{p_1 p_2 \cdots p_l} \left(1 - \frac{\ln p_1 p_2 \ldots p_l}{\ln x} \right) = g_l$$

for every fixed positive integer l. The desired relation

$$\mu_x(f_x(n) < u) \underset{x \to \infty}{\Rightarrow} F(u).$$

follows now from Lemma 2. This completes the proof of our theorem.

References

1. B.V. Levin, N.M. Timofeev, Распределение значений аддитивных функций, II, *Acta Arith.*, **33**, 327–351 (1977).
2. J.Šiaulys, Factorial moments of distributions of additive functions, *LMJ*, **40**(4), 389–408 (2000).
3. J.Šiaulys, On the logarithmic frequency of the values of additive functions, *Liet. Matem. Rink.*, **41** (spec. issue), 134–139 (2001).
4. P.D.T.A. Elliott, *Probabilistic Number Theory*, II, Spriger-Verlag, Berlin, Heidelberg, New York (1980).

REZIUMĖ

J. Šiaulys. Pirminių skaičių indikatorių sumų skirstiniai skirtingų dažnių atžvilgiu

Darbe charakterizuotos stipriai adityviųjų funkcijų sekos, kurių ribiniai skirstiniai įprasto ir logaritminio dažnio atžvilgiu sutampa. Nagrinėjamos stipriai adityviosios funkcijos su reikšmėmis pirminiuose skaičiuose lygiomis nuliui arba vienetui.