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Universality of Dirichlet L-functions with shifted characters

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Abstract. We study Dirichlet L-functions with shifted characters $L(s,\chi,H)=\sum_{n=1}^{\infty}\frac{\chi(n+H)}{n^s}$, where H is an integer. Here we obtain a joint universality theorem for such functions.

Keywords: Dirichlet L-functions, universality theorem.

1. Introduction

For Dirichlet L-functions we have the joint universality theorem (Lemma 2, see below) which says that a finite collection of Dirichlet L-functions with nonequivalent characters can approximate a finite collection of any analytic *nonvanishing* functions. In [1] we considered Dirichlet L-functions with shifted characters

$$L(s,\chi,H)=\sum_{n=1}^{\infty}\frac{\chi(n+H)}{n^s},$$

where χ is a Dirichlet character and H is an integer. Note that this function can be expressed by Hurwitz zeta-functions, also by Dirichlet L-functions:

$$\begin{aligned} L(s,\chi,H) &= \frac{1}{q^s}\sum_{k=1}^{q-1}\chi(k+H)\zeta\left(s,\frac{k}{q}\right)+\frac{1}{q^s}\chi(H)\zeta(s) \\ &= \frac{1}{q-1}\sum_{n=1}^{q-1}\eta_n(\chi,H)L(s,\chi_n)+\frac{1}{q^s}\chi(H)\zeta(s), \qquad (1) \end{aligned}$$

where χ_n, $n=1,\dots,q-1$, are all Dirichlet characters $\bmod q$ and

$$\eta_n(\chi,H)=\sum_{k=0}^{q-1}\chi(k+H)\bar{\chi}_n(k)\quad (n=1,\dots,q-1). \qquad (2)$$

From this we see that $L(s,\chi,H)$ can be continued meromorphically to the whole complex plane with at most one simple pole at $s=1$. For this function, in general, the Riemann hypothesis is not true. Let $q>3$ be prime and q not divide H. Then in [1] we with Steuding showed that for any σ_1 and σ_2 with $1/2<\sigma_1<\sigma_2<1$, there exists a constant $c = c(H, \sigma_1, \sigma_2) > 0$ such that for sufficiently large T the function $L(s, \chi, H)$ has more than cT zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$, $|t| < T$.

Here we prove a kind of joint universality theorem for $L(s, \chi, H)$.

THEOREM 1. *Let $0 < r < \frac{1}{4}$ and $H_j \in \mathbb{Z}$, $j = 1, \ldots, m$. Let $3 < q_1 < \ldots < q_m$ be primes and $\chi_j \bmod q_j$, $j = 1, \ldots, m$ be Dirichlet characters. Let $f_1(s), \ldots, f_m(s)$ be any continuous functions on the disc $|s| \leqslant r$, analytic in the interior of this disc, and if q_j divides H_j, let $f_j(s)$ also be nonvanishing. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{2T} \text{meas}\left\{\tau \in [-T, T],\ \max_{1\leqslant j\leqslant m} \max_{|s|\leqslant r} \left| L\left(s + \frac{3}{4} + i\tau, \chi_j, H_j\right) - f_j(s)\right| < \epsilon\right\} > 0.$$

Here and further meas$\{A\}$ denotes the Lebesgue measure of the set A. We say that two characters are equivalent if they are generated by the same primitive character. Note, that characters $\chi_1, \ldots, \chi_n$ from Theorem 1 are pairwise not equivalent. The proof of Theorem 1 makes use of the following Voronin theorem on the joint universality of the Dirichlet L-functions:

LEMMA 2 (Voronin [2]). *Let $0 < r < \frac{1}{4}$. Let $\chi_1, \ldots, \chi_m$ be pairwise non equivalent Dirichlet characters. Let $f_1(s), \ldots, f_m(s)$ be any nonvanishing continuous functions on the disc $|s| \leqslant r$, and analytic in the interior of this disc. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{2T} \text{meas}\left\{\tau \in [-T, T],\ \max_{1\leqslant j\leqslant m} \max_{|s|\leqslant r} \left| L\left(s + \frac{3}{4} + i\tau, \chi_j\right) - f_j(s)\right| < \epsilon\right\} > 0.$$

Also we need some information on coefficients $\eta_n(H)$. In [1] the following property was proved.

LEMMA 3. *Let $q > 3$ be a prime and q not divide H. Then there exist at least two different nonprincipal characters, say χ_i and χ_j, such that the numbers $\eta_i(H)$ and $\eta_j(H)$ are nonzero.*

Proof of Theorem 1. If χ_1 denotes the principal character mod q_j, then we have $\zeta(s) = \left(1 - \frac{1}{q_j^s}\right)^{-1} L(s, \chi_1)$. Thus, in view of (1), for j such that q_j not divides H_j, we have

$$\begin{aligned} L(s, \chi_j, H_j) &= \frac{1}{q_j - 1} \sum_{n=2}^{q_j-1} \eta_n(\chi_j, H_j) L(s, \chi_n) \\ &\quad + \left(\frac{\eta_1(\chi_j, H_j)}{q_j - 1} + \frac{\chi_j(H_j)}{q_j^s}\left(1 - \frac{1}{q_j^s}\right)^{-1}\right) L(s, \chi_1) \\ &= \frac{1}{q_j - 1} \sum_{n=2}^{q_j-1} \eta_n(\chi_j, H_j) L(s, \chi_n) \end{aligned}$$

$$+\frac{1}{q_j-1}\Big(\eta_1(H_j)+\frac{q_j-1}{q_j^s-1}\chi_j(H_j)\Big)L(s,\chi_1). \qquad (3)$$

By Lemma 3 we can suppose that $\eta_2(\chi_j, H_j)$ and $\eta_3(\chi_j, H_j)$ are nonzero. Let

$$M := 1 + \max_{\substack{|s|\le r\\ j=1,\dots,m}} |f_j(s)|.$$

Then $f_j(s) + M \neq 0$. For such j we define

$$f_{2,j}(s) := \frac{f_j(s)+M}{\eta_2(\chi_j, H_j)},$$

$$f_{3,j}(s) := -\frac{M}{\eta_3(\chi_j, H_j)},$$

$$f_{1,j}(s) := f_{4,j}(s) = \ldots = f_{q-1,j}(q) = \delta > 0$$

and for j such that q_j divides H_j (in this case $f_j \neq 0$),

$$f_{n,j}(s) := f_j(s),$$

where the constant $\delta > 0$ will be chosen later. Note that for j such that q_j divides H_j, we have $L(s, \chi_j, H_j) = L(s, \chi_j)$. By Lemma 2,

$$\liminf_{T\to\infty}\frac{1}{2T}\text{meas}\Big\{\tau\in[-T,T],\ \max_{1\leqslant j\leqslant m}\ \max_{1\leqslant n\leqslant q_j-1}\ \max_{|s|\leqslant r}\Big|L\Big(s+\frac{3}{4}+i\tau,\chi_n(\text{mod}\, q_j)\Big)$$
$$- f_{n,j}(s)\Big| < \delta\Big\} > 0.$$

For $|s - 3/4| \le r < 1/4$ and $\tau \in \mathbb{R}$ the quantity

$$\frac{q_j-1}{q_j^{s+i\tau}-1}$$

is bounded. By this, choosing sufficiently small δ, we obtain the theorem.

References

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2. S.M. Voronin, *Analytical Properties of the Dirichlet Generating Series of Arithmetical Objects*, Doctoral thesis, Moscow (1977) (in Russian).

REZIUMĖ

R. Garunkštis. Dirichlet L-funkcijų su paslinktais charakteriais universalumas

Mes studijuojame Dirichlet L-funkcijas su paslinktais charakteriais $L(s,\chi,H)=\sum_{n=1}^{\infty}\frac{\chi(n+H)}{n^s}$, čia H yra sveikasis skaičius. Šiame darbe įrodome daugiamatę universalumo teoremą šioms funkcijoms.