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A weighted universality theorem for zeta-functions of elliptic curves

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Let E: $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$, be an elliptic curve. Suppose that the discriminant $\Delta = -16(a^3 + 27b^2) \neq 0$. In this case an elliptic curve is non-singular.

Let, for a prime p, $\nu(p)$ be the number of solutions of the congruence

$$y^2 \equiv x^3 + ax + b \pmod{p},$$

and $\lambda(p) = p - \nu(p)$. Then the classical result of H. Hasse asserts that

$$|\lambda(p)| \leqslant 2\sqrt{p}$$

for each prime p.

Let $s = \sigma + it$ be a complex variable. To the curve E we attach the L-function $L_E(s)$ defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

Now it is known that $L_E(s)$ is analytically continuable to an entire function.

The papers [2], [3], [5] are devoted to the universality of the function $L_E(s)$. For example, in [2], [5] the following statement is given. Let $\nu_T(\ldots) = \frac{1}{T}\text{meas}\{\tau \in [0, T]\colon \ldots\}$, were in place of dots a condition satisfied by τ is to be written.

THEOREM A. *Let K be a compact subset of the strip $D = \{s \in \mathbb{C}\colon\ 1 < \sigma < \frac{3}{2}\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K} \left|L_E(s + i\tau) - f(s)\right| < \varepsilon\right) > 0.$$

The paper [3] contains a discrete version of Theorem A.

The aim of this note is to obtain a weighted universality theorem for the function $L_E(s)$.

Let T_0 be a fixed positive number, and let $w(\tau)$ be a positive function of bounded variation on $[T_0, \infty)$. Set

$$U = U(T, w) = \int_{T_0}^{T} w(\tau)\,\mathrm{d}\tau,$$

and suppose that $\lim_{T\to\infty} U(T,w)=+\infty$. Moreover, we need some weighted analogue of the Birkhoff-Khinchine theorem. Denote by E_ξ the mean of the random element ξ. Let $X(\tau,\omega)$, $\tau\in\mathbb{R}$, be an ergodic process defined on a certain probability space, $E|X(\tau,\omega)|<\infty$, with sample paths almost surely integrable in the Riemann sense over every finite interval. Suppose that the function $w(\tau)$ satisfies

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)X(t+\tau,\omega)\,\mathrm{d}\tau = EX(0,\omega)+\mathrm{o}(1+|t|)^{\delta} \tag{1}$$

almost surely for all $t\in\mathbb{R}$ with some $\delta>0$ as $T\to\infty$.

Denote by I_A the indicator function of the set A.

THEOREM 1. *Suppose that condition* (1) *is satisfied. Let* K *and* $f(s)$ *be the same as in Theorem A. Then, for every* $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{U}\int_{T_0}^{T} w(\tau)I_{\{\tau:\ \sup_{s\in K}|L_E(s+i\tau)-f(s)|<\varepsilon\}}\,\mathrm{d}\tau>0.$$

Clearly, Theorem A is a partial case of Theorem 1.

The proof of Theorem 1 is based on a limit theorem in the sense of the weak convergence of probability measures in the space of analytic functions for the function $L_E(s)$. Let G be a region in the complex plane. Denote by $H(G)$ the space of functions analytic on G, equipped with the topology of uniform convergence on compacta. Let, for $V>0$, $D_V=\{s\in\mathbb{C}\colon 1<\sigma<\frac{1}{2}, |t|<V\}$. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and define the probability measure

$$P_T(A)=\frac{1}{U}\int_{T_0}^{T} w(\tau)I_{\{\tau:\ L_E(s+i\tau)\in A\}}\,\mathrm{d}\tau,\qquad A\in\mathcal{B}(H(D_V)).$$

Denote by γ the unit circle $\gamma=\{s\in\mathbb{C}\colon |s|=1\}$ on the complex plane, and let

$$\Omega=\prod_p \gamma_p,$$

where $\gamma_p=\gamma$ for all primes p. With the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological group. Therefore there exists the probability Haar measure m_H on $(\Omega,\mathcal{B}(\Omega))$. Thus we obtain a probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Let $\omega(p)$ stand for the projection of $\omega\in\Omega$ into the coordinate space γ_p, and on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$ define the $H(D_V)$-valued random element $L_E(s,\omega)$ by the formula

$$L_E(s,\omega)=\prod_{p|\Delta}\Big(1-\frac{\lambda(p)\omega(p)}{p^s}\Big)^{-1}\prod_{p\nmid\Delta}\Big(1-\frac{\lambda(p)\omega(p)}{p^s}+\frac{\omega^2(p)}{p^{2s-1}}\Big)^{-1},\quad \omega\in\Omega. \tag{2}$$

LEMMA 2. *Under condition* (1) *the probability measure* P_T *converges weakly to the distribution of the random element* $L_E(s,\omega)$ *as* $T\to\infty$.

Proof. It is not difficult to see that, for $\sigma > \frac{3}{2}$,

$$L_E(s) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\alpha(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\lambda(p) = \alpha(p) + \beta(p)$, and

$$|\alpha(p)| \leqslant \sqrt{p}, \quad |\beta(p)| \leqslant \sqrt{p}.$$

Therefore, $L_E(s)$ is the Matsumoto zeta-function [7], [4] with parameters $\alpha = 0$ and $\beta = \frac{1}{2}$. Moreover, since by [1] every L-function attached to a non-singular elliptic curve over the field of rational numbers is the L-function attached to a certain newform of weight 2 of some congruence subgroup, we have that $L_E(s)$ is an entire function, and, for $\sigma > 1$,

$$L(\sigma + it) = \mathrm{O}(|t|^{c_1}), \quad |t| \geqslant t_0, \ c_1 > 0,$$

and [6]

$$\int_0^T |L(\sigma + it)|^2 \, \mathrm{d}t = \mathrm{O}(T), \quad T \to \infty.$$

Therefore, by Theorem 8 of [4] the probability measure

$$\frac{1}{U}\int_{T_0}^T w(\tau) I_{\{\tau:\ L_E(s+i\tau)\in A\}} \, \mathrm{d}\tau, \quad A \in \mathcal{B}(H(D)),$$

where $D_1 = \{s \in \mathbb{C}\colon \sigma > 1\}$, converges weakly to the distribution of the $H(D)$-valued random element defined by (2) as $T \to \infty$. Since the function $u\colon H(D_1) \to H(D)$ defined by the coordinatewise restriction is continuous, hence the lemma follows.

For the proof of Theorem 1 the support of the limit measure in Lemma 2 is needed. We recall that the support of a probability measure P defined on $(S, \mathcal{B}(S))$ is a minimal closed set $S_P \subset S$ such that $P(S_P) = 1$. The support S_P consists of all $x \in S$ such that for every neighbourhood G of x the inequality $P(G) > 0$ is satisfied.

Denote the limit measure in Lemma 2 by P_{L_E}, i.e., P_{L_E} is the distribution of the random element $L_E(s, \omega)$.

LEMMA 3. *The support of the measure P_{L_E} is the set*

$$S_V = \{g \in H(D_V)\colon g(s) \neq 0 \ \ or \ \ g(s) \equiv 0\}.$$

Proof. The proof of the lemma coincides with that of Lemma 8 in [5]. A sketch of the proof is also given in [3], Lemma 5. A more general statement for the Matsumoto zeta-function under some additional condition is contained in [4], Lemma 6.

Proof of Theorem 1. Let K be a compact subset of the strip D with connected complement. Then there exists $V > 0$ such that $K \subset D_V$.

First we suppose that the function $f(s)$ has a non-vanishing continuation to the rectangle D_V. Let G be the set of functions $g \in H(D_V)$ such that

$$\sup_{s\in K} |g(s) - f(s)| < \varepsilon.$$

Clearly, the set G is open. Moreover, by Lemma 3 we have that $G \subset S_V$. It is well known that the probability measure P_n converges weakly to P (P_n and P are given on $(S, \mathcal{B}(S))$) if and only if

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G) \tag{3}$$

for all open sets G of S. Therefore, in view of Lemmas 2 and 3

$$\liminf_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\ \sup_{s\in K} |L_E(s+i\tau)-f(s)|<\varepsilon\}}\, d\tau \geqslant P(G) > 0,$$

and in this case the theorem is proved.

The general case is reduced to the above proved case. Let $f(s)$ be the same as in the statement of Theorem 1. Then by the Mergelyan theorem, see, for example, [8], there exists a sequence of polynomials $\{p_n(s)\}$ such that $p_n(s) \to f(s)$ as $n \to \infty$ uniformly on K. Since $f(s) \neq 0$ on K, there exists a sufficiently large n_0 such that $p_{n_0}(s) \neq 0$ on K and

$$\sup_{s\in K} |f(s) - p_{n_0}(s)| < \frac{\varepsilon}{4}.$$

However, the polynomial $p_{n_0}(s)$ has only finitely many zeros, and therefore we can find a region G with connected complement such that $K \subset G$ and $p_{n_0}(s) \neq 0$ on G. Hence, we can choose a continuous version of $\log p_{n_0}(s)$ on G which is analytic in the interior of G. By the Mergelyan theorem again we can find a polynomial $q_n(s)$ such that

$$\sup_{s\in K} \left|p_{n_0}(s) - e^{q_n(s)}\right| < \frac{\varepsilon}{4}.$$

The latter two inequalities show that

$$\sup_{s\in K} \left|f(s) - e^{q_n(s)}\right| < \frac{\varepsilon}{2}. \tag{4}$$

Since, clearly, $e^{q_n(s)} \neq 0$, the first part of the proof implies

$$\liminf_{T\to\infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\{\tau:\ \sup_{s\in K} |L_E(s+i\tau)-e^{q_n(s)}|<\frac{\varepsilon}{2}\}}\, d\tau > 0.$$

This and inequality (3) yield the assertion of the theorem.

For example, we can take $w(\tau) = \tau^{-1}$.

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REZIUMĖ

V. Garbaliauskienė. Ribinė teorema su svoriu elipsinių kreivių dzeta funkcijoms

Gauta universalumo teorema su svoriu elipsinės kreivės L-funkcijai.