Liet. matem. rink, **46**, *spec. nr.*, 2006, 445–450

Invariance principle for independent random variables with infinite variance

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Abstract. A functional central limit theorem for self-normalized adaptive process $U_{m,N}^{-1}\zeta_n$ is considered, where $U_{m,N}$ is a sum of squares of block-sums of size m, as m and the number of blocks $N = n/m$ tend to infinity.

Keywords: adaptive process, domain of attraction, normal law, series scheme.

1. Introduction and results

Various partial sums processes can be built from the sums $S_n = \varepsilon_1 + \cdots + \varepsilon_n$ of independent identically distributed mean zero random variables. The white noise sequence $\varepsilon_i, i \geqslant 1$ belongs to the domain of attraction of the normal law (denoted $\varepsilon_1 \in \mathrm{DAN}$), that is, it is not required that variance of ε_1 is finite. The attention of this paper is focused on the so called adaptive partial sums process, denoted ζ_n. Set $U_0 = 0$ and define

$$U_{m,k}^2 = \sum_{j=1}^{k} (S_{jm} - S_{(j-1)m})^2, \quad k = 1, \ldots, N, \ \ 1 \leqslant m < n, \tag{1}$$

where $N = [n/m]$ and $[a]$ denotes the integer part of a.

Adaptive means that the vertices of the corresponding random polygonal line (denoted ζ_n) have their abscissas at the random points $U_{m,k}^2/U_{m,N}^2$ instead of the deterministic equispaced points k/N. By this construction the slope of each line adapts itself to the value of the corresponding sum of random variables. The ζ_n process formally is defined on $[0,1]$ by linear interpolation between the vertices $(U_{m,k}^2/U_{m,N}^2, S_{mk})$, $k = 1, \ldots, N$. The main result of the paper is the following

THEOREM 1. *Assume that* $\varepsilon_1 \in \mathrm{DAN}$. *Then*

$$U_{m,N}^{-1}\zeta_n \xrightarrow[n\to\infty]{\mathcal{D}} W \ \ \textit{in the space} \quad C[0,1]$$

provided $m = m(n) \to \infty$ *and* $m/n \to 0$ *as* $n \to \infty$.

Observe that the case $m = const$ is proved in the article [5], and developed in [3]. This Theorem concerns m tending to infinity and a block-wise constructed polygonal line.

2. Proofs

The proof of the Theorem 1 follows from the Lemma formulated below. Denote $V_k^2 = \sum_{j=1}^k \varepsilon_j^2$, $k = 1, \dots, n$ and $Y_j := \sum_{k=1}^m \epsilon_{m(j-1)+k}$, $j = 1, \dots, N$. Y's are mean zero, independent random variables. Since $U_{m,N}(V_n)^{-1} \xrightarrow[n\to\infty]{\mathrm{P}} 1$ (see [2]), it follows that $(U_{m,N})^{-1}\sum_{j=1}^N Y_j$ weakly converges to $N(0,1)$. The next lemma shows the validity of O'Brien's type convergence if ε's and n is replaced by Y's and N.

LEMMA 2. *Let $E\varepsilon_1 = 0$ and $\varepsilon_1 \in DAN$ with normalizing constants $b_n := l_n\sqrt{n}$. Then for each positive δ the following relations hold true:*

$$NP\big(|Y_1| > \delta l_n\sqrt{n}\big) \to 0, \tag{2}$$

$$Nb_n^{-2}\boldsymbol{E}(Y_1)^2 I\big(|Y_1| \leqslant \delta l_n\sqrt{n}\big) \to 1, \tag{3}$$

$$Nb_n^{-1}\boldsymbol{E}Y_1 I\big(|Y_1| \leqslant \delta l_n\sqrt{n}\big) \to 0, \tag{4}$$

$$m \to \infty, \quad m/n \to 0.$$

Proof. Denote for each $\tau > 0$

$$R_j^1 := \varepsilon_j I\big(|\varepsilon_j| \leqslant b_n\tau\big), \quad R_j^2 := \varepsilon_j I\big(|\varepsilon_j| > b_n\tau\big), \quad j = 1, \dots, n.$$

Next for each positive δ:

$$P\big(|Y_1| \geqslant b_n\delta\big) \leqslant P\big(|R_1^1 + \dots + R_m^1| \geqslant b_n\delta/2\big) + P\big(|R_1^2 + \dots + R_m^2| \geqslant b_n\delta/2\big) := A_1 + A_2.$$

Now from the inclusion $(|R_1^2 + \dots + R_m^2| \geqslant b_n\delta/2) \subset (\bigcup_{i=1}^m \{|\varepsilon_i| > b_n\tau\})$ one gets

$$NA_2 \leqslant N \cdot mP\big(|\varepsilon_1| > b_n\tau\big) = \mathrm{o}(1).$$

From Chebyshev's inequality it follows

$$P\big(|R_1^1 + \dots R_m^1| \geqslant b_n\delta/2\big) \leqslant \frac{16E|R_1^1 + \dots R_m^1|^4}{b_n^4\delta^4} \leqslant B_1 + B_2,$$

where

$$B_1 := \frac{32}{b_n^4\delta^4}\Big(mE(R_1^1)^4 + m(m-1)\big(E(R_1^1)^2\big)^2\Big), \quad B_2 := \frac{32}{b_n^4\delta^4}E\bigg(\sum_{i\neq j}^m R_i^1 R_j^1\bigg)^2.$$

Since $E(R_1^1)^4 \leqslant b_n^2\tau^2 E(R_1^1)^2$, it follows

$$NB_1 \leqslant 32\tau^2\frac{n}{b_n^2\delta^4}E(R_1^1)^2 + \frac{32n(m-1)}{b_n^4\delta^4}\big(E(R_1^1)^2\big)^2 = \frac{32\tau^2}{\delta^4}\Big(\frac{n}{b_n^2}E(R_1^1)^2\Big) + \frac{32(m-1)}{n\delta^4}\Big(\frac{n}{b_n^2}E(R_1^1)^2\Big)^2.$$

Now letting first $n \to \infty$ $(m/n \to 0)$ and then $\tau \to 0$ one gets that $NB_1 = \mathrm{o}(1)$.

Next

$$B_2 = \frac{32}{b_n^4 \delta^4}\Big(m(m-1)\big(E(R_1^1)^2\big)^2 + m(m-1)(m-2)(ER_1^1)^2 E(R_1^1)^2 + m(m-1)(m-2)(m-4)(ER_1^1)^4 \Big).$$

Now

$$NB_2 = \frac{32(m-1)}{n\delta^4}\Big(\frac{n}{b_n^2}E(R_1^1)^2\Big)^2 + \frac{32}{b_n^2\delta^4}\cdot\frac{(m-1)(m-2)}{n^2}\cdot\Big(\frac{n}{b_n^2}E(R_1^1)^2\Big)(nER_1^1)^2 + \frac{32}{b_n^4\delta^4}\frac{(m-1)(m-2)(m-3)}{n^3}(nER_1^1)^4.$$

Thus it follows that $NB_2 = \mathrm{o}(1)$. The proof of (2) is complete.

Now observe that for each positive δ

$$P\Big(\max_{1\leqslant j\leqslant N}|Y_j| \geqslant b_n\delta\Big) = P\Big(\bigcup_{j=1}^{N}\{|Y_j| \geqslant b_n\delta\}\Big) \leqslant NP\big(|Y_1| \geqslant b_n\delta\big).$$

Thus convergence (2) implies

$$\frac{\max_{1\leqslant j\leqslant N}|Y_j|}{b_n} \xrightarrow[n\to\infty]{\mathrm{P}} 0. \tag{5}$$

Now we have that independent variables (Y_j/b_n), $j = 1, \dots, N$ are uniformly vanishing. Denote

$$Y_i^* = b_n^{-1}\Big(Y_i^\varepsilon - \mathbf{E}Y_1 I\big(|Y_1| \leqslant b_n\delta\big)\Big), \quad i = 1, \dots, N,$$

$$r_n := N\mathbf{E}\frac{Y_1^*}{1+(Y_1^*)^2} + \frac{N}{b_n}\mathbf{E}Y_1 I\big(|Y_1| \leqslant b_n\delta\big).$$

Also for all real x define

$$\Xi_n(x) := N\mathbf{E}\frac{(Y_1^*)^2}{1+(Y_1^*)^2} I\big(|Y_1^*| \leqslant x\big).$$

Observe that $(Y_1 + \cdots + Y_N)/b_n$ weakly converges to the standard normal law if and only if

$$\Xi_n(x) \to I(x > 0), \quad r_n \to 0. \tag{6}$$

Note that first convergence in (6) is full.

Next we have that

$$N P\big(|Y_1^*| > \delta\big) \leqslant N \frac{1+\delta^2}{\delta^2} \mathbf{E} \frac{(Y_1^*)^2}{1+(Y_1^*)^2} I\big(|Y_1^*| > \delta\big)$$
$$= N \frac{1+\delta^2}{\delta^2} \Big(\mathbf{E} \frac{(Y_1^*)^2}{1+(Y_1^*)^2} - \mathbf{E} \frac{(Y_1^*)^2}{1+(Y_1^*)^2} I\big(|Y_1^*| \leqslant \delta\big)\Big).$$

From convergence (6) we deduce

$$N P\big(|Y_1^*| > \delta\big) \to 0, \quad n \to \infty. \tag{7}$$

Now we prove (4). To this aim observe that from the right hand side convergence in (6) it suffices to prove

$$N \mathbf{E} \frac{Y_1^*}{1+(Y_1^*)^2} \to 0.$$

Denote $\delta_1 := 2\delta + 1$ and lets split

$$N \mathbf{E} \frac{Y_1^*}{1+(Y_1^*)^2} = N \mathbf{E} Y_1^* I\big(|Y_1^*| < \delta_1\big) - N \mathbf{E} \frac{(Y_1^*)^3}{1+(Y_1^*)^2} I\big(|Y_1^*| < \delta_1\big)$$
$$+ N \mathbf{E} \frac{Y_1^*}{1+(Y_1^*)^2} I\big(|Y_1^*| \geqslant \delta_1\big) := I_1 - I_2 + I_3.$$

Now from (6)

$$I_2 = \int_{|x|<\delta_1} x \,\mathrm{d}\Xi_n(x) \to \int_{|x|<\delta_1} x \,\mathrm{d}I(x>0) = 0,$$
$$I_3 = \int_{|x|\geqslant\delta_1} \frac{1}{x} \,\mathrm{d}\Xi_n(x) \to \int_{|x|\geqslant\delta_1} \frac{1}{x} \,\mathrm{d}I(x>0) = 0.$$

Denote $c_n(\delta) := b_n^{-1} \mathbf{E} Y_1^{\varepsilon} I(|Y_1^{\varepsilon}| \leqslant b_n\delta)$ and observe that $|c_n(\delta)| \leqslant \delta$.

$$I_1 = N \mathbf{E} Y_1^* I\big(|Y_1^*| < \delta_1\big) = N \mathbf{E}\big(b_n^{-1} Y_1 - c_n(\delta)\big)\Big\{ I\big(|Y_1| < b_n\delta\big)$$
$$- I\big(|Y_1| < b_n\delta, |b_n^{-1}Y_1 - c_n(\delta)| \geqslant \delta_1\big) + I\big(|Y_1| \geqslant b_n\delta, |b_n^{-1}Y_1 - c_n(\delta)| < \delta_1\big)\Big\}.$$

First observe

$$|I_1| \leqslant N\delta P\big(|Y_1| \geqslant b_n\delta\big) + 2\delta N P\big(|Y_1^*| \geqslant \delta_1\big) + \delta_1 N P\big(|Y_1| \geqslant b_n\delta\big),$$

and $P(|Y_1^*| \geqslant \delta_1) \leqslant P(|Y_1| \geqslant b_n\delta)$. Now applying (2) one gets $|I_1| \to 0$. The proof of convergence (4) is complete.

Finally we prove (3). For each $0 < \tau < \delta$

$$0 \leqslant N\mathbf{E}(Y_1^*)^2 I\big(|Y_1^*| \leqslant \delta\big) - N\mathbf{E}\frac{(Y_1^*)^2}{1+(Y_1^*)^2} I\big(|Y_1^*| \leqslant \delta\big)$$
$$= N\mathbf{E}\frac{(Y_1^*)^4}{1+(Y_1^*)^2} I\big(|Y_1^*| \leqslant \delta\big)$$
$$\leqslant N\tau^2\mathbf{E}\frac{(Y_1^*)^2}{1+(Y_1^*)^2} I\big(|Y_1^*| \leqslant \tau\big) + \delta^4 NP\big(|Y_1^*| > \tau\big).$$

From (7) one gets $NP(|Y_1^*| > \tau) = o_n(1)$. Now applying (6) one get's

$$\limsup_{n\to\infty}\Big(N\mathbf{E}(Y_1^*)^2 I\big(|Y_1^*| \leqslant \delta\big) - N\mathbf{E}\frac{(Y_1^*)^2}{1+(Y_1^*)^2} I\big(|Y_1^*| \leqslant \delta\big)\Big) = \tau^2.$$

Thus letting $\tau \to 0$ it follows $N\mathbf{E}(Y_1^*)^2 I(|Y_1^*| \leqslant \delta) \to 1$. Now consider

$$J := N\Big(\mathbf{E}(Y_1^*)^2 I\big(|Y_1^*| \leqslant \delta\big) - b_n^{-2}\mathbf{E}(Y_1)^2 I\big(|Y_1| \leqslant \delta b_n\big)\Big)$$
$$= Nb_n^{-2}\mathbf{E}(Y_1)^2\Big\{I\big(|b_n^{-1}Y_1 - c_n(\delta)| \leqslant \delta\big) - I\big(b_n^{-1}|Y_1| \leqslant \delta\big)\Big\}$$
$$- 2Nb_n^{-1}c_n(\delta)\mathbf{E}Y_1 I\big(|b_n^{-1}Y_1 - c_n(\delta)| \leqslant \delta\big) + N(c_n(\delta))^2 P\big(|b_n^{-1}Y_1 - c_n(\delta)| \leqslant \delta\big)$$
$$:= J_1 - J_2 + J_3.$$

Observe that by (2)

$$J_1 = Nb_n^{-2}\mathbf{E}(Y_1)^2\Big\{I\big(|b_n^{-1}Y_1 - c_n(\delta)| \leqslant \delta, b_n^{-1}|Y_1| > \delta\big)$$
$$- I\big(|b_n^{-1}Y_1 - c_n(\delta)| > \delta, b_n^{-1}|Y_1| \leqslant \delta\big)\Big\}$$
$$\leqslant 4N\delta^2 P(b_n^{-1}|Y_1| > \delta) \to 0, \quad n \to \infty.$$

Next by (4)

$$|J_2| \leqslant 2c_n(\delta)Nb_n^{-1}\mathbf{E}|Y_1| I\big(|Y_1| \leqslant 2\delta b_n\big) \leqslant 4\delta Nc_n(\delta) \to 0, \quad n \to \infty$$

and $|J_3| \leqslant N^{-1}(Nc_n(\delta))^2 \to 0$, $n \to \infty$. Thus $J \to 0$ and the proof of convergence (3) is complete.

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REZIUMĖ

M. Juodis. Invariantiškumo principas nepriklausomiems atsitiktiniams dydžiams su begaline dispersija

Darbe įrodoma funkcinė centrinė ribinė teorema serijų schemai. Nagrinėjamas adaptuotas procesas su auto-normavimu.