Liet. matem. rink, **46**, *spec. nr.*, 2006, 238–241

Sequent calculus for hybrid logic

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This article deals with pure hybrid logic formulas, i.e., formulas, where atomic formulas are only nominals. Various methods are known for derivation of tautologies. In C. Areces, H. de Nivelle, M. de Rijke works [1] resolution method, in P. Blackburn, M. Marx work [3] – tableau method, in T. Braüner work [4] – natural deduction and sequent calculus, in P. Blackburn [2] and J. Seligman [5] works – sequent calculus are defined.

This article deals with formulas containing $\vee$, & and $\neg$ operator only before nominals. Also recall that operators @, $\downarrow$ are self duals (see [2]). Only sequents of the form $\Gamma \vdash$ will be considered, i.e., formulas will appear on the left side of the $\vdash$ symbol. Therefore we will not use $\vdash$ symbol. This article presents calculus *H*, where sequents with an empty succedent are derivable and which contains only two rules for nominals. We will show that any formula *F* is derivable in calculus *H* iff $F \vdash$ is derivable in sequent calculus described in T. Braüner work [4].

Sequent calculus H

Axioms: $\Gamma, @_s t, @_s \neg t$ $\quad$ $\Gamma, @_s \neg s$

Rules:

$$\frac{\Gamma}{\Gamma, F} \qquad (\&)\ \frac{\Gamma, @_s F, @_s G}{\Gamma, @_s(F\&G)} \qquad (\vee)\ \frac{\Gamma, @_s F \quad \Gamma, @_s G}{\Gamma, @_s(F \vee G)}$$

$$(\Diamond_i)\ \frac{\Gamma, @_s\Diamond_i t, @_t F}{\Gamma, @_s \Diamond_i F},\ \text{t is new} \qquad (\Box_i)\ \frac{\Gamma, @_t F, @_s \Box_i F, @_s\Diamond_i t}{\Gamma, @_s\Box_i F, @_s \Diamond_i t}$$

$$(Simp)\ \frac{\Gamma, @_t F}{\Gamma, @_s @_t F} \qquad (Sub)\ \frac{\Gamma[t/s]}{\Gamma, @_s t} \qquad (\downarrow)\ \frac{\Gamma, @_s F[s/t]}{\Gamma, @_s \downarrow t F}$$

Γ denotes a finite set of formulas (possibly empty). *F, G* are formulas, *s, t* – nominals.

Calculus *Hp* is a modified version of the calculus, presented in T. Braüner article [4], where symbol : is replaced by @ and nominal names are changed to those used in calculus *H*. Only rules with $\forall$ operator are not included in the calculus *Hp*.

Axiom: $\Gamma, F \vdash \Delta, F$

Rules:

$$(\& \vdash)\ \frac{\Gamma, @_s F, @_s G \vdash \Delta}{\Gamma, @_s(F\&G) \vdash \Delta} \qquad (\vdash \&)\ \frac{\Gamma \vdash \Delta, @_s F \quad \Gamma \vdash \Delta, @_s G}{\Gamma \vdash \Delta, @_s(F\&G)}$$

$$(\to\vdash)\ \frac{\Gamma \vdash \Delta, @_s F \qquad \Gamma, @_s G \vdash \Delta}{\Gamma, @_s(F \to G) \vdash \Delta} \qquad (\vdash\to)\ \frac{\Gamma, @_s F \vdash \Delta, @_s G}{\Gamma \vdash \Delta, @_s(F \to G)}$$

$$(\Box_i \vdash)\ \frac{\Gamma \vdash \Delta, @_s \Diamond_i t \qquad \Gamma, @_t F \vdash \Delta}{\Gamma, @_s \Box_i F \vdash \Delta} \qquad (\vdash \Box_i)\ \frac{\Gamma, @_s \Diamond_i t \vdash \Delta, @_t F}{\Gamma \vdash \Delta, @_s \Box_i F}$$

$$(@ \vdash)\ \frac{\Gamma, @_t F \vdash \Delta}{\Gamma, @_s @_t F \vdash \Delta} \qquad (\vdash @)\ \frac{\Gamma \vdash \Delta, @_t F}{\Gamma \vdash \Delta, @_s @_t F}$$

$$(\downarrow\vdash)\ \frac{\Gamma \vdash \Delta, @_s w \qquad \Gamma, @_w F[w/t] \vdash \Delta}{\Gamma, @_s \downarrow t F \vdash \Delta} \qquad (\vdash\downarrow)\ \frac{\Gamma, @_s w \vdash \Delta, @_w F[w/t]}{\Gamma \vdash \Delta, @_s \downarrow t F}$$

$$(Ref)\ \frac{\Gamma, @_s s \vdash \Delta}{\Gamma \vdash \Delta} \qquad (Nom1)\ \frac{\Gamma \vdash \Delta, @_t s \qquad \Gamma \vdash \Delta, @_t F}{\Gamma \vdash \Delta, @_s F}$$

$$(Nom2)\ \frac{\Gamma \vdash \Delta, @_s t \qquad \Gamma \vdash \Delta, @_s \Diamond_i w \qquad \Gamma, @_t \Diamond_i w \vdash \Delta}{\Gamma \vdash \Delta}$$

On a *Hp* basis we build a new calculus *Hpm*. All rules remain the same except rules $(\vdash\to)$ and $(\to\vdash)$. Those rules we remove from a rule list and add these rules to the calculus:

$$(\vdash \vee)\ \frac{\Gamma \vdash \Delta, @_s F, @_s G}{\Gamma \vdash \Delta, @_s(F \vee G)} \quad \text{and} \quad (\vdash \neg)\ \frac{\Gamma, @_s F \vdash \Delta}{\Gamma \vdash \Delta, @_s \neg F}$$

$$(\vee \vdash)\ \frac{\Gamma, @_s F \vdash \Delta \qquad \Gamma, @_s G \vdash \Delta}{\Gamma, @_s(F \vee G) \vdash \Delta} \quad \text{and} \quad (\neg \vdash)\ \frac{\Gamma \vdash \Delta, @_s F}{\Gamma, @_s \neg F \vdash \Delta}$$

We also add one more axiom:

$$\Gamma, F, \neg F \vdash \Delta \quad (N)$$

LEMMA 1. *Sequent is derivable in calculus Hp iff it is derivable in calculus Hpm.*

Proof. Suppose a derivable sequent $\Gamma \vdash \Delta$ and its derivation tree in calculus *Hp* is given. In in the given sequent formulas of the form $F \to G$ replace with formulas $\neg F \vee G$ and applications of a rule $(\vdash\to)$ replace with a straight application of rules $(\vdash \vee)$ and $(\vdash \neg)$. Similarly applications of a rule $(\to\vdash\ )$ are replaced with straight application of rules $(\vee \vdash)$ and $(\neg \vdash)$.

Axiom (N) is included for simplicity purposes because when a sequent Γ, $@_s F$, $@_s \neg F \vdash \Delta$ is given an axiom is obtained in one step by applying a rule $(\neg \vdash)$.

THEOREM 1. *Sequent is derivable in calculus Hpm iff it is derivable in calculus H.*

Proof. We will show how to replace application of one calculus rules by application of another calculus rules.

It is obvious that rules (&) and ($\& \vdash$), ($\vee$) and ($\vee \vdash$), ($Simp$) and ($@ \vdash$) are equivalent, i.e., by applying these rules in correspondent calculus we obtain the same sequents.

We will not replace the application of a rule (Ref) in calculus *Hpm* with any rule in calculus *H* because the application of this rule is meaningful only to sequent of the form $\Gamma \vdash @_s s$. This sequent is derivable in calculus *Hpm* iff sequent $\Gamma, @_s \neg s \vdash$ is derivable. Sequent $\Gamma, @_s \neg s$ is an axiom in calculus *H*.

Now we will note what rules can be applied in calculus *Hpm*. Since in the given sequent formulas can appear only on the left side of the $\vdash$ and negation stands straight before nominals, after application of any rules on the right side of the derivation symbol we can obtain only formulas of the form $@_s t$ or $@_s \Diamond_i t$, where t is nominal. Therefore rules ($\vdash \&$), ($\vdash \Box_i$), ($\vdash @$), ($\vdash\downarrow$), ($\vdash \vee$) , ($\vdash \neg$) can not be applied. Likewise we will not apply rule ($\neg \vdash$) because of an axiom (N). Besides to those formulas of the form $@_s t$ or $@_s \Diamond_i t$ only rules ($Nom1$) and ($Nom2$) can be applied. These rules can be brought up to the top of derivation tree. That is why we will show how rules ($Nom1$), ($Nom2$), ($\Box_i \vdash$), ($\downarrow\vdash$) are replaced in calculus *Hpm*, when sequents with formulas on the right side of de rivation symbol are axioms:

- ($Nom1$)

$$\frac{\Gamma, @_w s, @_w t \vdash @_w s \quad \Gamma, @_w s, @_w t \vdash @_w t}{\Gamma, @_w s, @_w t \vdash @_s t} \quad \Rightarrow \quad \frac{\Gamma, @_w s, @_s t, @_s \neg t}{\Gamma, @_w s, @_w t, @_s \neg t}(Sub)$$

and

$$\frac{\Gamma, @_w s, @_w \Diamond_i t \vdash @_w s \quad \Gamma, @_w s, @_w \Diamond_i t \vdash @_w \Diamond_i t}{\Gamma, @_w s, @_w \Diamond_i t \vdash @_s \Diamond_i t}$$

$$\Downarrow$$

$$\cfrac{\cfrac{\cfrac{\Gamma, @_w s, @_s \Diamond_i t, @_s \Box_i \neg t, @_s \neg t, @_s t, @_s \Diamond_i s}{\Gamma, @_w s, @_s \Diamond_i t, @_s \Box_i \neg t, @_s \neg t}(\Diamond_i)}{\Gamma, @_w s, @_s \Diamond_i t, @_s \Box_i \neg t}(\Box_i)}{\Gamma, @_w s, @_w \Diamond_i t, @_s \Box_i \neg t}(Sub)$$

- ($Nom2$)

$$\frac{\Gamma, @_s t \vdash @_s t \quad \Gamma, @_s \Diamond_i w \vdash @_s \Diamond_i w \quad \Gamma, @_t \Diamond_i w \vdash @_t \Diamond_i w}{\Gamma, @_s t, @_s \Diamond_i w \vdash @_t \Diamond_i w}$$

$$\Downarrow$$

$$\cfrac{\cfrac{\Gamma, @_s t, @_t \Diamond_i w, @_t \Box_i \neg w, @_w \neg w}{\Gamma, @_s t, @_t \Diamond_i w, @_t \Box_i \neg w}(\Box_i)}{\Gamma, @_s t, @_s \Diamond_i w, @_t \Box_i \neg w}(Sub)$$

- $(\Box_i \vdash)$

$$\frac{\Gamma,\ @_s\Diamond_i t \vdash @_s\Diamond_i t \qquad \Gamma,\ @_t F,\ @_s\Diamond_i t \vdash}{\Gamma,\ @_s\Box_i F,\ @_s\Diamond_i t \vdash} \quad \Longrightarrow \quad \frac{\Gamma,\ @_t F,\ @_s\Box_i F,\ @_s\Diamond_i t}{\Gamma,\ @_s\Box_i F,\ @_s\Diamond_i t}$$

- $(\downarrow\vdash)$

$$\frac{\Gamma,\ @_s w \vdash \Delta,\ @_s w \qquad \Gamma,\ @_w F[w/t] \vdash \Delta}{\Gamma,\ @_s w,\ @_s \downarrow t F \vdash \Delta} \quad \Longrightarrow \quad \cfrac{\cfrac{\Gamma,\ @_s w,\ @_w F[w/s]}{\Gamma,\ @_s w,\ @_s F[s/t]}(Sub)}{\Gamma,\ @_s w,\ @_s \downarrow t F}(\downarrow)$$

Next we will analyse rules of a calculus *H* $(\Diamond_i)$, $(\Box_i)$, (Sub) and $(\downarrow)$. Application of a rule $(\Box_i)$ is replaced by application of a rule $(\Box_i \vdash)$, the left sequent becomes an axiom in calculus *Hpm* (see above). Let's analyse rule $(\Diamond_i)$. Since $\Diamond_i F \equiv \neg\Box_i\neg F$ then in calculus *Hpm* we first perform such a replacement and then apply rules $(\neg \vdash)$, $(\vdash \Box_i)$, $(\vdash \neg)$.

Application of a rule $(\downarrow)$ is replaced like this:

$$\frac{\Gamma,\ @_s F[s/t]}{\Gamma,\ @_s \downarrow t F} \quad \Longrightarrow \quad \cfrac{\cfrac{\Gamma,\ @_s s \vdash @_s s}{\Gamma \vdash @_s s}(Ref) \qquad \Gamma,\ @_s F[s/t] \vdash}{\Gamma,\ @_s \downarrow t F \vdash}(\downarrow\vdash)$$

The rule (Sub) replaces one variable symbol by another. In calculus *Hpm* this function is performed by rules $(\Box_i \vdash)$, $(\downarrow\vdash)$, $(Nom1)$, $(Nom2)$.

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REZIUMĖ

S. Norgėla, A. Šalaviejienė. Sekvencinis skaičiavimas hibridinei logikai

Nagrinėjamos grynosios hibridinės logikos formulės, kuriose neigimas yra tik prieš nominalus. Aprašytas sekvencinis skaičiavimas su tuščiu sukcedentu, kuriame tėra tik dvi taisyklės nominalams.