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Asymptotic analysis of Sturm–Liouville problem with Robin and two-point boundary conditions

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Abstract. We analyze the initial value problem and get asymptotic expansions for solution. We investigate the characteristic equation for Sturm–Liouville problem with one classical Robin type boundary condition and another two-point nonlocal boundary condition. Finally, we obtain asymptotic expansions for eigenvalues and eigenfunctions.

Keywords: Sturm–Liouville problem; Robin condition; two-point nonlocal conditions; asymptotics of eigenvalues and eigenfunctions

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Introduction

Consider the following one-dimensional Sturm–Liouville equation

$$-u''(t) + q(t)u(t) = \lambda u(t), \quad t \in [0,1], \tag{1}$$

where the real-valued function $q \in C[0,1]$; $\lambda = s^2$ is a complex spectral parameter and $s = x + \imath y$; $x, y \in \mathbb{R}$. We will use the notation $Q(t) = \frac{1}{2}\int_0^t q(\tau)d\tau$.

In this article $s \in \mathbb{C}_s := \mathbb{R}_s \cup \mathbb{C}_s^+ \cup \mathbb{C}_s^-$, where $\mathbb{R}_s := \mathbb{R}_s^- \cup \mathbb{R}_s^+ \cup \mathbb{R}_s^0$, $\mathbb{R}_s^- := \{s = x + \imath y \in \mathbb{C} : x = 0, y > 0\}$, $\mathbb{R}_s^+ := \{s = x + \imath y \in \mathbb{C} : x > 0, y = 0\}$, $\mathbb{R}_s^0 := \{s = 0\}$, $\mathbb{C}_s^+ := \{s = x + \imath y \in \mathbb{C} : x > 0, y > 0\}$ and $\mathbb{C}_s^- := \{s = x + \imath y \in \mathbb{C} :\ x > 0,\ y < 0\}$. Then a map $\lambda = s^2$ is the bijection between $\mathbb{C}_s$ and $\mathbb{C}_\lambda := \mathbb{C}$.

We shall investigate Sturm–Liouville Problem (SLP) which consist of equation (1) on $[0,1]$ with one classical (local) Robin type Boundary Condition (BC)

$$\cos\alpha\ u(0) + \sin\alpha\ u'(0) = 0, \quad \alpha \in (0, \pi), \tag{2}$$

and another two-point Nonlocal Boundary Condition (NBC)

$$\text{(Case 1)} \qquad u'(1) = \gamma u(\xi), \qquad \xi \in [0,1], \tag{3_1}$$

$$\text{(Case 2)} \qquad u'(1) = \gamma u'(\xi), \qquad \xi \in [0,1), \tag{3_2}$$

$$\text{(Case 3)} \qquad u(1) = \gamma u(\xi), \qquad \xi \in [0,1), \tag{3_3}$$

where $\gamma \in \mathbb{R}$. We consider the Dirichlet and the Neumann BC:

$$\text{(Case d)} \qquad u(0) = 0, \tag{4_d}$$

$$\text{(Case n)} \qquad u'(0) = 0, \tag{4_n}$$

too. The Sturm–Liouville problem (1), (4_d), (3_3) was investigated in [2], the Sturm–Liouville problem (1), (4_n), (3) was investigated in [4].

1 Asymptotic expansions for Initial Value Problem

In this section we present some statements about solution of IVP. These statements were proved in [3]. We will use them for investigation asymptotic expansions for SLP (1)–(3). Additionally, we introduce some notation related to our asymptotical analysis of this problem.

Let $\lambda = s^2$, $s \in \mathbb{C}_s$ and $\omega_{\alpha s}(t)$ be a solution of equation (1) satisfying the initial conditions

$$\omega_{\alpha s}(0) = \sin\alpha, \qquad \omega'_{\alpha s}(0) = -\cos\alpha. \tag{5}$$

The function $\omega(t,s,\alpha) = \omega_{\alpha s}(t)$ is an analytic (holomorphic) function of s and this function satisfies boudary condition (2). We denote $\varphi_s(t) := \omega_{0s}(t) = \omega(t,s,0)$ and $\psi_s(t) := \omega_{\pi/2,s}(t) = \omega(t,s,\pi/2)$.

Under the condition that $q \in C^r[0,1]$, $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, asymptotic expansions may be obtained for $\varphi_s(t)$ [3] and $\psi_s(t)$ [4]. We will use recursive formula

$$p^0_{i+1}(t) = -\frac{1}{2}\int_0^t q(\tau)p^0_i(\tau)\,d\tau - \sum_{j=2+\varrho}^{i} \frac{(qp^0_{j-1})^{(i-j)}(t) + (-1)^i(qp^0_{j-1})^{(i-j)}(0)}{2^{i-j+2}}. \tag{6}$$

Lemma 1. (See [3, Lemma 7]) *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geqslant q_0$ we have the asymptotic expansions*

$$(\varphi_s)^{(l)}_s(t,s) = -\sum_{j=1}^{r+1} p^l_j(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + \mathcal{O}\big(s^{-(r+2)}e^{(r+2)|y|t}\big), \tag{7}$$

$$(\varphi'_s)^{(l)}_s(t,s) = -\sum_{j=0}^{r} \bar{p}^l_j(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + \mathcal{O}\big(s^{-(r+1)}e^{(r+2)|y|t}\big) \tag{8}$$

for $l \in \mathbb{N}_0$, where $p^k_1(t) = -tp^{k-1}_1(t)$, $p^k_i(t) = (1-i)p^{k-1}_{i-1}(t) - tp^{k-1}_i(t)$, $i = \overline{2, r+1}$, $\bar{p}^k_0(t) = -t\bar{p}^{k-1}_0(t)$, $\bar{p}^k_i(t) = (1-i)\bar{p}^{k-1}_{i-1}(t) - t\bar{p}^{k-1}_i(t)$, $i = \overline{1,r}$, $k \in \mathbb{N}$, $\bar{p}^0_i(t) = p^{0\prime}_i(t) - p^0_{i+1}(t)$, $i = \overline{1,r}$, $\bar{p}^0_0(t) = 1$, and $p^0_j(t)$ is calculated by (6) for $i = \overline{1,r}$ with $p^0_1(t) = -1$ and $\varrho = 0$.

Lemma 2. (See [4, Lemma 9].) *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geqslant q_0$ we have the asymptotic expansions*

$$(\psi_s)_s^{(l)}(t,s) = -\sum_{j=0}^{r} p_j^l(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + \mathcal{O}\big(s^{-(r+1)}e^{(r+2)|y|t}\big), \tag{9}$$

$$(\psi'_s)_s^{(l)}(t,s) = -\sum_{j=-1}^{r-1} \bar{p}_j^l(t)\cos\Big(st + \frac{\pi}{2}(j-l)\Big)s^{-j} + \mathcal{O}\big(s^{-r}e^{(r+2)|y|t}\big) \tag{10}$$

for $l \in \mathbb{N}_0$, where $p_0^k(t) = -tp_0^{k-1}(t)$, $p_i^k(t) = (1-i)p_{i-1}^{k-1}(t) - tp_i^{k-1}(t)$, $i = \overline{1,r}$, $\bar{p}_{-1}^k(t) = -t\bar{p}_{-1}^{k-1}(t)$, $\bar{p}_i^k(t) = (1-i)\bar{p}_{i-1}^{k-1}(t) - t\bar{p}_i^{k-1}(t)$, $i = \overline{0,r-1}$, $k \in \mathbb{N}$, $\bar{p}_i^0(t) = p_i^{0\prime}(t) - p_{i+1}^0(t)$, $i = \overline{0,r-1}$, $\bar{p}_{-1}^0(t) = 1$, and $p_j^0(t)$ is calculated by (6) for $i = \overline{0,r-1}$ with $p_0^0(t) = -1$ and $\varrho = -1$.

Remark 1. (See [3, Lemma 7], [4, Lemma 7].) In the case $q \in C[0,1]$ and $l = 0$ we have the asymptotic expansions:

$$\varphi_s(t) = -\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}e^{|y|t}\big), \qquad \psi_s(t) = \cos(st) + \mathcal{O}\big(s^{-1}e^{|y|t}\big),$$
$$\varphi'_s(t) = -\cos(st) + \mathcal{O}\big(s^{-1}e^{|y|t}\big), \qquad \psi'_s(t) = -s\sin(st) + \mathcal{O}\big(e^{|y|t}\big).$$

Remark 2. In the case $q \in C^1[0,1]$ and $l = 0$ we have the asymptotic expansions

$$\begin{aligned}
\varphi_s(t) &= -\sin(st)s^{-1} + Q(t)\cos(st)s^{-2} + \mathcal{O}\big(s^{-3}e^{3|y|t}\big),\\
\psi_s(t) &= \cos(st) + Q(t)\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}e^{3|y|t}\big),\\
\varphi'_s(t) &= -\cos(st) - Q(t)\sin(st)s^{-1} + \mathcal{O}\big(s^{-2}e^{3|y|t}\big),\\
\psi'_s(t) &= -s\sin(st) + Q(t)\cos(st) + \mathcal{O}\big(s^{-1}e^{3|y|t}\big),
\end{aligned}$$

where $Q(t) = \frac{1}{2}\int_0^t q(\tau)d\tau$.

We can calculate the first functions p_j^l and $\bar{p}_j^l$ in Case d (for function φ):

$$\begin{aligned}
&p_1^0 = -1, && p_2^0 = Q(t), && p_3^0 = -\tfrac{1}{2}(Q(t))^2 + \tfrac{1}{4}q(t) + \tfrac{1}{4}q(0),\\
&p_1^1 = t, && p_2^1 = 1 - tQ(t), && p_1^2 = -t^2,\\
&\bar{p}_0^0 = 1, && \bar{p}_1^0 = -Q(t), && \bar{p}_2^0 = \tfrac{1}{2}(Q(t))^2 + \tfrac{1}{4}q(t) - \tfrac{1}{4}q(0),\\
&\bar{p}_0^1 = -t, && \bar{p}_1^1 = tQ(t), && \bar{p}_0^2 = t^2;
\end{aligned}$$

and in Case n (for function ψ):

$$\begin{aligned}
&p_0^0 = -1, && p_1^0 = Q(t), && p_2^0 = -\tfrac{1}{2}(Q(t))^2 + \tfrac{1}{4}q(t) - \tfrac{1}{4}q(0),\\
&p_0^1 = t, && p_1^1 = -tQ(t), && p_0^2 = -t^2,\\
&\bar{p}_{-1}^0 = 1, && \bar{p}_0^0 = -Q(t), && \bar{p}_1^0 = \tfrac{1}{2}(Q(t))^2 + \tfrac{1}{4}q(t) + \tfrac{1}{4}q(0),\\
&\bar{p}_{-1}^1 = -t, && \bar{p}_0^1 = 1 + tQ(t), && \bar{p}_{-1}^2 = t^2.
\end{aligned}$$

We will use an additional index to distinguish cases: $p_j^{\mathrm{d},l}(t)$, $\bar{p}_j^{\mathrm{d},l}(t)$ (in Case d), $p_j^{\mathrm{n},l}(t)$, $\bar{p}_j^{\mathrm{n},l}(t)$ (in Case n).

The following integral equation holds [1, 5]:

$$\omega_{\alpha s}(t) - \frac{1}{s}\int_0^t q(\tau)\sin\bigl(s(t-\tau)\bigr)\omega_{\alpha s}(\tau)d\tau = \sin\alpha\cos(st) - \cos\alpha\frac{\sin(st)}{s}. \tag{11}$$

If $\alpha = 0$, then the solution of this integral equation is φ_s, if $\alpha = \pi/2$, then the solution of this integral equation is ψ_s. By superposition principle we have

$$\omega_{\alpha s}(t) = \cos\alpha\cdot\varphi_s(t) + \sin\alpha\cdot\psi_s(t). \tag{12}$$

So, we get asymptotic expansions for function $\omega_{\alpha s}$.

Lemma 3. *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geqslant q_0$ we have the asymptotic expansions*

$$(\omega_{\alpha s})_s^{(l)}(t,s) = -\sum_{j=0}^{r} p_j^l(t)\cos\biggl(st + \frac{\pi}{2}(j-l)\biggr)s^{-j} + \mathcal{O}\bigl(s^{-(r+1)}e^{(r+2)|y|t}\bigr), \tag{13}$$

$$(\omega'_{\alpha s})_s^{(l)}(t,s) = -\sum_{j=-1}^{r-1} \bar{p}_j^l(t)\cos\biggl(st + \frac{\pi}{2}(j-l)\biggr)s^{-j} + \mathcal{O}\bigl(s^{-r}e^{(r+2)|y|t}\bigr) \tag{14}$$

for $l \in \mathbb{N}_0$, where

$$p_0^l(t) = \sin\alpha\cdot p_0^{\mathrm{n},l}(t), \quad p_j^l(t) = \cos\alpha\cdot p_j^{\mathrm{d},l}(t) + \sin\alpha\cdot p_j^{\mathrm{n},l}(t), \quad j = \overline{1,r}, \tag{15}$$

$$\bar{p}_{-1}^l(t) = \sin\alpha\cdot\bar{p}_{-1}^{\mathrm{n},l}(t), \quad \bar{p}_j^l(t) = \cos\alpha\cdot\bar{p}_j^{\mathrm{d},l}(t) + \sin\alpha\cdot\bar{p}_j^{\mathrm{n},l}(t), \quad j = \overline{0,r-1}. \tag{16}$$

Remark 3. Formulas (15)–(16) are valid for $\alpha = 0$, but in (7)–(8) we have more accurate the asymptotic expansions with

$$p_j^{\mathrm{d},l}(t)\cos\Bigl(st + \frac{\pi}{2}(r+1-l)\Bigr)s^{-(r+1)} + \mathcal{O}\bigl(s^{-(r+2)}e^{(r+2)|y|t}\bigr),$$
$$\bar{p}_j^{\mathrm{d},l}(t)\cos\Bigl(st + \frac{\pi}{2}(r-l)\Bigr)s^{-r} + \mathcal{O}\bigl(s^{-(r+1)}e^{(r+1)|y|t}\bigr),$$

instead $\mathcal{O}\bigl(s^{-(r+1)}e^{(r+2)|y|t}\bigr)$ and $\mathcal{O}\bigl(s^{-r}e^{(r+2)|y|t}\bigr)$ in (13)–(14).

Corollary 1. *If $q \in C[0,1]$ and $\alpha \in (0,\pi)$, then we have asymptotic expansions:*

$$\omega_{\alpha s}(t) = \sin\alpha\cdot\cos(st) + \mathcal{O}\bigl(s^{-1}e^{|y|t}\bigr), \quad \omega'_{\alpha s}(t) = -\sin\alpha\cdot\sin(st)\,s + \mathcal{O}(e^{|y|t}).$$

Corollary 2. *If $q \in C^1[0,1]$ and $\alpha \in (0,\pi)$, then we have asymptotic expansions:*

$$\omega_{\alpha s}(t) = \sin\alpha\cdot\cos(st) + \bigl(-\cos\alpha + \sin\alpha Q(t)\bigr)\sin(st)s^{-1} + \mathcal{O}\bigl(s^{-2}e^{3|y|t}\bigr),$$
$$\omega'_{\alpha s}(t) = -\sin\alpha\cdot\sin(st)\,s + \bigl(-\cos\alpha + \sin\alpha Q(t)\bigr)\cos(st) + \mathcal{O}\bigl(s^{-1}e^{3|y|t}\bigr).$$

Lemma 4. *Let $x \in \mathbb{R}_s^+$, $\delta \in \mathbb{R}$, $q \in C^r[0,1]$, $Q_j(x)$, $j = \overline{1,r}$ are bounded functions. If $s = x + \delta$,*

$$\delta = \sum_{j=1}^{r} Q_j(x)x^{-j} + \mathcal{O}\bigl(x^{-(r+1)}\bigr),$$

then we have the following asymptotic expansions

$$\omega_{\alpha s}(t) = \sum_{j=0}^{r} R_j(t,x)x^{-j} + \mathcal{O}\big(x^{-(r+1)}\big), \quad \omega'_{\alpha s}(t) = \sum_{j=-1}^{r-1} \bar{R}_j(t,x)x^{-j} + \mathcal{O}\big(x^{-r}\big),$$

where

$$R_0(t,x) = \sin\alpha \cdot R_0^{\mathrm{n}}(t,x), \qquad R_j(t,x) = \cos\alpha \cdot R_j^{\mathrm{d}}(t,x) + \sin\alpha \cdot R_j^{\mathrm{n}}(t,x),\ j = \overline{1,r},$$
$$\bar{R}_{-1}(t,x) = \sin\alpha \cdot \bar{R}_{-1}^{\mathrm{n}}(t,x), \quad \bar{R}_j(t,x) = \cos\alpha \cdot \bar{R}_j^{\mathrm{d}}(t,x) + \sin\alpha \cdot \bar{R}_j^{\mathrm{n}}(t,x),\ j = \overline{0,r-1},$$

and functions

$$R_{m+1}^{\mathrm{d}}(t,x) = -\sum_{\substack{n_1+\ldots+n_m=l,\ j\geqslant 1,\\ j+n_1+2n_2+\ldots+mn_m=m+1}} \tfrac{1}{n_1!\cdots n_m!} p_j^{\mathrm{d},l}(t)\cos\big(xt + \tfrac{\pi}{2}(j-l)\big)Q_1^{n_1}(x)\cdots Q_m^{n_m}(x),$$
$$\bar{R}_m^{\mathrm{d}}(t,x) = -\sum_{\substack{n_1+\ldots+n_m=l,\ j\geqslant 0,\\ j+n_1+2n_2+\ldots+mn_m=m}} \tfrac{1}{n_1!\cdots n_m!} \bar{p}_j^{\mathrm{d},l}(t)\cos\big(xt + \tfrac{\pi}{2}(j-l)\big)Q_1^{n_1}(x)\cdots Q_m^{n_m}(x),$$
$$R_m^{\mathrm{n}}(t,x) = -\sum_{\substack{n_1+\ldots+n_m=l,\ j\geqslant 0,\\ j+n_1+2n_2+\ldots+mn_m=m}} \tfrac{1}{n_1!\cdots n_m!} p_j^{\mathrm{n},l}(t)\cos\big(xt + \tfrac{\pi}{2}(j-l)\big)Q_1^{n_1}(x)\cdots Q_m^{n_m}(x),$$
$$\bar{R}_{m-1}^{\mathrm{n}}(t,x) = -\sum_{\substack{n_1+\ldots+n_m=l,\ j\geqslant -1,\\ j+n_1+2n_2+\ldots+mn_m=m-1}} \tfrac{1}{n_1!\cdots n_m!} \bar{p}_j^{\mathrm{n},l}(t)\cos\big(xt + \tfrac{\pi}{2}(j-l)\big)Q_1^{n_1}(x)\cdots Q_m^{n_m}(x),$$

$m = \overline{0,r}$.

Proof. The proof follows from (12) and asymptotic expansions for $\varphi_s(t)$ [3, Corollary 2] and $\psi_s(t)$ [4, Corollary 2]. □

Remark 4. In the case $\alpha = 0$ we have more accurate the asymptotic expansions

$$\varphi_s(t) = \sum_{j=1}^{r+1} R_j^{\mathrm{d}}(t,x)x^{-j} + \mathcal{O}\big(x^{-(r+2)}\big), \quad \varphi'_s(t) = \sum_{j=0}^{r} \bar{R}_j^{\mathrm{d}}(t,x)x^{-j} + \mathcal{O}\big(x^{-(r+1)}\big).$$

Corollary 3. *If $q \in C[0,1]$ and $\alpha \in (0,\pi)$, then we have asymptotic expansions:*

$$\omega_{\alpha s}(t) = \sin\alpha \cdot \cos(xt) + \mathcal{O}\big(x^{-1}\big), \quad \omega'_{\alpha s}(t) = -\sin\alpha \cdot \sin(xt)\,x + \mathcal{O}(1).$$

Corollary 4. *If $q \in C^1[0,1]$ and $\alpha \in (0,\pi)$, then we have asymptotic expansions:*

$$\omega_{\alpha s}(t) = \sin\alpha \cdot \cos(xt) + \big(-\cos\alpha + \sin\alpha(Q(t) - tQ_1(x))\big)\sin(xt)x^{-1} + \mathcal{O}\big(x^{-2}\big),$$
$$\omega'_{\alpha s}(t) = -\sin\alpha \cdot \sin(xt)\,x + \big(-\cos\alpha + \sin\alpha(Q(t) - tQ_1(x))\big)\cos(xt) + \mathcal{O}\big(x^{-1}\big).$$

2 Asymptotic expansions for characteristic equations

Substituting $\omega_{\alpha s}(t)$ into (3) we get the characteristic equation

$$h_\alpha(s) := \omega'_{\alpha s}(1) - \gamma\omega_{\alpha s}(\xi) = 0, \tag{17_1}$$
$$h_\alpha(s) := \omega'_{\alpha s}(1) - \gamma\omega'_{\alpha s}(\xi) = 0, \tag{17_2}$$
$$h_\alpha(s) := \omega_{\alpha s}(1) - \gamma\omega_{\alpha s}(\xi) = 0. \tag{17_3}$$

Let's define functions:

$$h^l_{-1}(s) := -\bar p^l_{-1}(1)\sin\big(s-\tfrac{\pi}{2}l\big) = (-1)^{l-1}\sin\alpha\sin\big(s-\tfrac{\pi}{2}l\big),$$

$$h^l_j(s) := \gamma p^l_j(\xi)\cos\big(\xi s+\tfrac{\pi}{2}(j-l)\big) - \bar p^l_j(1)\cos\big(s+\tfrac{\pi}{2}(j-l)\big),\ j=\overline{0,r-1}, \quad (18_1)$$

$$h^l_j(s) := \gamma \bar p^l_j(\xi)\cos\big(\xi s+\tfrac{\pi}{2}(j-l)\big) - \bar p^l_j(1)\cos\big(s+\tfrac{\pi}{2}(j-l)\big),\ j=\overline{-1,r-1}, \quad (18_2)$$

$$h^l_j(s) := \gamma p^l_j(\xi)\cos\big(\xi s+\tfrac{\pi}{2}(j-l)\big) - p^l_j(1)\cos\big(s+\tfrac{\pi}{2}(j-l)\big),\ j=\overline{0,r}, \quad (18_3)$$

where functions p^l_j and $\bar p^l_j$ are defined by formulas (15)–(16). For example,

$$h^0_{-1}(s) = -\sin\alpha\sin s,\quad h^1_{-1}(s) = -\sin\alpha\cos s,$$

$$h^0_0(s) = \big(\sin\alpha\, Q(1)-\cos\alpha\big)\cos s - \sin\alpha\,\gamma\cos(\xi s), \quad (19_1)$$

$$h^0_{-1}(s) = -\sin\alpha\big(\sin s-\gamma\sin(\xi s)\big),\quad h^1_{-1}(s) = -\sin\alpha\big(\cos s-\gamma\xi\cos(\xi s)\big),$$

$$h^0_0(s) = \big(\sin\alpha\, Q(1)-\cos\alpha\big)\cos s - \big(\sin\alpha\, Q(\xi)-\cos\alpha\big)\gamma\cos(\xi s), \quad (19_2)$$

$$h^0_0(s) = \sin\alpha\big(\cos s-\gamma\cos(\xi s)\big),\quad h^1_0(s) = -\sin\alpha\big(\sin s-\gamma\xi\sin(\xi s)\big),$$

$$h^0_{-1}(s) = \big(\sin\alpha\, Q(1)-\cos\alpha\big)\sin s - \big(\sin\alpha\, Q(\xi)-\cos\alpha\big)\gamma\sin(\xi s). \quad (19_3)$$

We will use the notation: $\rho=-1$, $a_k := (k-1/2)\pi$ in Cases 1, 2; $\rho=0$, $a_k := (k-1)\pi$ in Case 3, $k\in\mathbb{N}$.

Lemma 5. *Suppose $|\gamma|<1$ in Cases 2 and 3. Then $|h^0_\rho(a_k+\imath y)| \geqslant \varkappa e^{|y|}$, $\varkappa>0$.*

Proof. In Case 1 we have

$$\begin{aligned}|h^0_{-1}(a_k+\imath y)| &= \sin\alpha|\sin(a_k+\imath y)| = \sin\alpha|\sin a_k\cosh y+\imath\cos a_k\sinh y)| \\ &= \sin\alpha\cosh y \geqslant \sin\alpha e^{|y|}/2.\end{aligned}$$

From inequalities

$$|\sin s-\gamma\sin(\xi s)| \geqslant (|\sin x|-|\gamma|\,|\sin(\xi x)|)\cosh y \geqslant (|\sin x|-|\gamma|)\cosh y, \quad (20_2)$$

$$|\cos s-\gamma\cos(\xi s)| \geqslant (|\cos x|-|\gamma|\,|\cos(\xi x)|)\cosh y \geqslant (|\cos x|-|\gamma|)\cosh y \quad (20_3)$$

we get (in Cases 2 and 3)

$$|h^0_\rho(a_k+\imath y)| \geqslant \sin\alpha(1-|\gamma|)\cosh y \geqslant \sin\alpha(1-|\gamma|)e^{|y|}/2. \qquad \square$$

Lemma 6. *Suppose $|\gamma|<1$ in Cases 2 and 3. There exists $B>0$ such that $|h^0_\rho(s)| \geqslant \kappa e^{|y|}$, $\kappa>0$ for $|y|\geqslant B$.*

Proof. We estimate

$$|\sin s-\gamma\sin(\xi s)| \geqslant |\sin s|-|\gamma|\,|\sin(\xi s)| \geqslant \sinh|y|-|\gamma|\cosh(\xi y),$$

$$|\cos s-\gamma\cos(\xi s)| \geqslant |\cos s|-|\gamma|\,|\cos(\xi s)| \geqslant \sinh|y|-|\gamma|\cosh(\xi y).$$

For $|\gamma|<1$ we have

$$\lim_{y\to+\infty}\big(\sinh y-|\gamma|\cosh(\xi y)\big)e^{-y} = \tfrac12(1-|\gamma|\cdot\lfloor\xi\rfloor) \geqslant \tfrac12(1-|\gamma|)>0.$$

So, in Cases 2 and 3 there exists $B>0$ such that

$$|h^0_\rho(s)| \geqslant \tfrac14\sin\alpha(1-|\gamma|)e^{|y|}.$$

for $|y|\geqslant B$. The proof in Case 1 repeats the proof in Case 2 with $\gamma=0$. $\square$

Lemma 7. *Let $s \in \mathbb{C}_s$ and $q \in C^r[0,1]$. Then for $|s| \geqslant q_0$ the asymptotic expansion*

$$h_\alpha^{(l)}(s) = \sum_{j=\rho}^{r+\rho} h_j^l(s) s^{-j} + \mathcal{O}\big(s^{-(r+1+\rho)} e^{(r+2)|y|}\big) \tag{21}$$

is valid, $l \in \mathbb{N}_0$.

Proof. The proof for $\alpha \in (0,\pi)$ is the same as in case $\alpha = \pi/2$ [4, see Lemma 11]. □

Remark 5. In the case $q \in C[0,1]$ we have (see Corollary 1) the asymptotic expansion

$$h_\alpha(s) = h_\rho^0(s) s^{-\rho} + \mathcal{O}\big(s^{-(1+\rho)} e^{|y|}\big). \tag{22}$$

Remark 6. If $\alpha = 0$ (the problem (1), (4_d), (3)) formula (21) is valid with $\rho = 0$ in Cases 1, 2; $\rho = 1$ in Case 3, and

$$h_0^0(s) = -\cos s, \quad h_0^1(s) = -\sin s, \tag{23_1}$$

$$h_0^0(s) = \gamma\cos(\xi s) - \cos s, \quad h_0^1(s) = \sin s - \gamma\xi\sin(\xi s), \tag{23_2}$$

$$h_1^0(s) = \gamma\sin(\xi s) - \sin s, \quad h_1^1(s) = \gamma\xi\cos(\xi s) - \cos s. \tag{23_3}$$

So, if $\alpha = 0$, then Lemma 5 and Lemma 6 are valid for the problem (1), (4_d), (3) with $\rho = 0$, $a_k = k\pi$, $k \in \mathbb{N}$, in Cases 1 and 2, $\rho = 1$, $a_k = (k - 1/2)\pi$, $k \in \mathbb{N}$, in Case 3.

Let us consider positive $s = x > 0$, $q \in C^r[0,1]$, $r \geqslant 1$. We investigate equation $h_\alpha(x+\delta) = 0$, $\delta \in \mathbb{R}$, with additional condition

$$|h_\rho^1(x)| \geqslant \varkappa > 0. \tag{24}$$

Lemma 8. *Suppose $|\gamma| < 1$ in Cases 2 and 3. If $h_\rho^0(x) = 0$, then (24) is valid. The constant $\varkappa$ is the same for all such x.*

Proof. In Case 1 if $h_{-1}^0(x) = -\sin\alpha\sin x = 0$, then $x = x_k = k\pi$, $k \in \mathbb{N}$ and $|h_{-1}^1(x_k)| = |-\sin\alpha\cos x_k| = \sin\alpha > 0$.

In Case 2 we have equation $h_{-1}^0(x) = -\sin\alpha\big(\sin x - \gamma\sin(\xi x)\big)$. If x_k are the root of equation $\sin x - \gamma\sin(\xi x) = 0$, then we get [2, see Lemma 4 and Corollary 3] that $|h_{-1}^1(x_k)| = \big|-\sin\alpha\big(\cos x_k - \gamma\xi\cos(\xi x_k)\big)\big| \geqslant \sin\alpha|\cos x_k - \gamma\xi\cos(\xi x_k)| \geqslant \sin\alpha\cdot\kappa = \varkappa > 0$.

In Case 3 we have equation $h_0^0(x) = \sin\alpha\big(\cos x - \gamma\cos(\xi x)\big)$. If x_k are the root of equation $\cos x - \gamma\cos(\xi x) = 0$, then we get [2, see Lemma 5] that $|h_0^1(x_k)| \geqslant \sin\alpha|\sin x_k - \gamma\xi\sin(\xi x_k)| \geqslant \sin\alpha\big(|\sin x_k| - |\gamma|\cdot|\sin(\xi x_k)|\big) \geqslant \sin\alpha\cdot\kappa = \varkappa > 0$. □

Remark 7. Lemma 4 and Lemma 5 in [2] were proved for $\xi \in (0,1)$, but it is easy to see, that they are valid for $\xi = 0$, too. So, Lemma 8 is proved for all ξ (see (3)).

Remark 8. If $\alpha = 0$, then Lemma 8 is valid with $\rho = 0$ in Cases 1 and 2, $\rho = 1$ in Case 3. The proof is the same.

Let's denote the function

$$Q_1(x) = -h_{1+\rho}^0(x)\big(h_\rho^1(x)\big)^{-1}.$$

If functions $Q_1,\dots,Q_{k-1}$ are defined, then we can find functions

$$z_l(x)=\sum_{\substack{n_1+\dots+n_{k-1}=i,\ j\geqslant 0,\\ j+n_1+2n_2+\dots+(k-1)n_{k-1}=l}} -h^{i+1}_{j+\rho}(x)(h^1_\rho(x))^{-1}\tfrac{Q_1^{n_1}(x)\cdots Q_{k-1}^{n_{k-1}}(x)}{(i+1)n_1!\cdots n_{k-1}!},$$

$l=\overline{1,k-1}$ and function

$$Q_k(x)=\sum_{\substack{n_1+\dots+n_{k-1}=l,\ j>0,\\ j+n_1+2n_2+\dots+(k-1)n_{k-1}=k}} -h^{0}_{j+\rho}(x)(h^1_\rho(x))^{-1}\tfrac{l!z_1^{n_1}(x)\cdots z_{k-1}^{n_{k-1}}(x)}{n_1!\cdots n_{k-1}!}.$$

If $q\in C^1[0,1]$, then

$$Q_1(x)=\frac{\bigl(\sin\alpha\,Q(1)-\cos\alpha\bigr)\cos x-\sin\alpha\,\gamma\cos(\xi x)}{\sin\alpha\cos x}, \tag{25_1}$$

$$Q_1(x)=\frac{\bigl(\sin\alpha\,Q(1)-\cos\alpha\bigr)\cos x-\bigl(\sin\alpha\,Q(\xi)-\cos\alpha\bigr)\gamma\cos(\xi x)}{\sin\alpha\bigl(\cos x-\gamma\xi\cos(\xi x)\bigr)}, \tag{25_2}$$

$$Q_1(x)=\frac{\bigl(\sin\alpha\,Q(1)-\cos\alpha\bigr)\sin x-\bigl(\sin\alpha\,Q(\xi)-\cos\alpha\bigr)\gamma\sin(\xi x)}{\sin\alpha\bigl(\sin x-\gamma\xi\sin(\xi x)\bigr)}. \tag{25_3}$$

Lemma 9. *If $q\in C^r[0,1]$ and $\delta=o(1)$, $h^0_\rho(x)=0$, then asymptotic expansion*

$$\delta=\sum_{j=1}^{r}Q_j(x)x^{-j}+\mathcal{O}\bigl(x^{-(r+1)}\bigr)$$

is valid, where $Q_j(x)$, $j=\overline{1,r}$, are bounded functions.

Proof. The proof can be found in [4, see proof of Lemma 14]. □

3 Spectral asymptotics for eigenvalues and eigenfunctions

In this section we assume, that $|\gamma|<1$ in Cases 2, 3. Let us denote domains $D_k=\{s\in\mathbb{C}\colon |x|\leqslant a_k,\ |y|\leqslant a_k\}$, $D_{sk}=\mathbb{C}_s\cap D_k$, $k\in\mathbb{N}$ ($k>1$ in Case 3), contours $\Gamma_{sk}=\mathbb{C}_s\cap\partial D_k$, and intervals $I_k:=\bigl(a_k,\,a_{k+1}\bigr)\subset D_{s,k+1}\setminus D_{sk}$, $k\in\mathbb{N}$.

Lemma 10. *Suppose $|\gamma|<1$ in Cases 2 and 3. If $q\in C[0,1]$, then it follows that the number of zeros of functions $h_\alpha(s)$ and $h^0_\rho(s)s^{-\rho}$ is the same inside Γ_{sk} for sufficiently large k.*

Proof. We have (see (22)) $h_\alpha(s)=h^0_\rho(s)s^{-\rho}+\mathcal{O}\bigl(s^{-1-\rho}e^{|y|}\bigr)$. Using Lemma 5 and Lemma 6 we estimate $|\mathcal{O}\bigl(s^{-1-\rho}e^{|y|}\bigr)|\leqslant c_1|s|^{-1-\rho}e^{|y|}<\min\{\kappa,\varkappa\}|s|^{-\rho}e^{|y|}\leqslant|h^0_\rho(s)s^{-\rho}|$ on the contours Γ_{sk} for sufficiently large k. Therefore, by Rouché theorem it follows that the number of zeros of $h_\alpha(s)$ and $h^0_\rho(s)s^{-\rho}$ are the same inside Γ_{sk} for sufficiently large k. □

Corollary 5. *If $q\in C[0,1]$, then it follows that the number of zeros of functions $h_\alpha(s)$ and $h^0_\rho(s)$ is the same between Γ_{sk} and $\Gamma_{s,k+1}$ for sufficiently large k.*

Remark 9. If $\alpha = 0$, then Lemma 10 and Corollary 5 are valid. The proof is the same. In Case 3 $s = 0$ isn't zero of the function $h_1^0(s)s^{-1} = \gamma s^{-1}\sin(\xi s) - s^{-1}\sin s$ for $|\gamma| < 1$.

From (19) (and (23) for $\alpha = 0$) we have that function h_ρ^0 has only one positive root $x_k \in I_k$, $k \in \mathbb{N}$. For example, $x_k = \pi k$ (and $x_k = \pi(k + 1/2)$ for $\alpha = 0$) in Case 1. In Cases 2 and 3 existence of such root follows from [4, Lemma 4 and Lemma 5]. Thus, function $h_\alpha(s)$ has only one root s_k between Γ_{sk} and $\Gamma_{s,k+1}$ for sufficiently large k.

In Case 2 ($\sin\alpha \neq 0$) $a_k = (k - 1/2)\pi$ and

$$h_\alpha(a_k) = -\sin\alpha\big(\sin a_k - \gamma\sin(\xi a_k)\big)a_k + \mathcal{O}(1) = a_k\sin\alpha\big((-1)^k + \gamma\sin(\xi a_k) + \mathcal{O}(k^{-1})\big).$$

If $|\gamma| < 1$, then $\operatorname{sign}\big((-1)^k + \gamma\sin(\xi a_k) + \mathcal{O}(k^{-1})\big) = (-1)^k$. So, we have

$$\operatorname{sign}\big(h_\alpha(a_k)h_\alpha(a_{k+1})\big) = -1$$

for sufficiently large k. This formula is valid in Cases 1 and 3. Moreover, it is valid for $\alpha = 0$. Then from Intermediate Value Theorem at least one root of the function $h_\alpha(s)$ lies in I_k for sufficiently large k. So, s_k is real root for such k.

We have $s_k \sim x_k \sim \pi k$ (as $k \to \infty$). Then $h_\alpha(s_k)\cdot s_k^\rho = h_\rho^0(s_k) + \mathcal{O}\big(k^{-1}\big) = 0$ and $\lim_{k\to\infty} h_\rho^0(s_k) = 0$. The function h_ρ^0 is analytic and has one root in I_k. Additionally, $|h_\rho^1(x_k)| \geqslant \kappa > 0$ (see Lemma 8). Therefore, $s_k \to x_k$ as $k \to \infty$ or

$$s_k = x_k + o(1) \quad (\text{as } k \to \infty). \tag{26}$$

Now we will investigate the distribution of these positive eigenvalues of problem (1)–(3), and we leave out the note about sufficiently large k.

Let us denote $\delta_k = s_k - x_k$. We have that $\delta_k = o(1)$.

Theorem 1. *Let $q \in C^r[0,1]$. For eigenvalues $\lambda_k = s_k^2$ and eigenfunctions u_k of problem (1)–(3), we have the asymptotic expansions*

$$s_k = x_k + \sum_{j=1}^{r} Q_j(x_k)x_k^{-j} + \mathcal{O}\big(k^{-(r+1)}\big), \tag{27}$$

$$u_k(t) = \sum_{j=0}^{r} R_j(t, x_k)x_k^{-j} + \mathcal{O}\big(k^{-(r+1)}\big) \tag{28}$$

for sufficiently large k.

Proof. We have $\delta_k = o(1)$. So, all conditions of Lemma 9 are valid, and it follows (27). Then we apply Corollary 1 and get (28). □

Corollary 6. *Let $q \in C[0,1]$. For eigenvalues $\lambda_k = s_k^2$ and eigenfunctions u_k of problem (1)–(3), the asymptotic expansions*

$$s_k = x_k + \mathcal{O}\big(k^{-1}\big), \qquad u_k(t) = R_0(t, x_k) + \mathcal{O}\big(k^{-1}\big)$$

are valid for sufficiently large k, where $R_0(t,x) = \sin\alpha\cos(xt)$.

Corollary 7. *If $q \in C^1[0,1]$, then the asymptotic expansions*

$$s_k = x_k + Q_1(x_k)x_k^{-1} + \mathcal{O}(k^{-2}),$$
$$u_k(t) = R_0(t,x_k) + R_1(t,x_k)x_k^{-1} + \mathcal{O}(k^{-2})$$

are valid for sufficiently large k, where $Q_1(x)$ is defined by (25)*,* $R_0(t,x) = \sin\alpha\cos(xt)$*,* $R_1(t,x) = (-\cos\alpha + \sin\alpha(Q(t) - tQ_1(x)))\sin(xt)$*.*

Remark 10. (See [2].) If $\alpha = 0$, then formula (27) and asymptotic expansion

$$u_k(t) = \sum_{j=1}^{r+1} R_j(t,x_k)x_k^{-j} + \mathcal{O}(k^{-(r+2)}) \tag{29}$$

are valid for sufficiently large k. If $q \in C[0,1]$, then $R_1(t,x) = -\sin(xt)$. If $q \in C^1[0,1]$, then $R_2(t,x) = (Q(t) - tQ_1(x)))\cos(xt)$ and

$$Q_1(x) = \frac{Q(1)\sin x - \gamma\sin(\xi x)}{\sin x}, \tag{30$_1$}$$
$$Q_1(x) = \frac{Q(1)\sin x - \gamma Q(\xi)\sin(\xi x)}{\sin x - \gamma\xi\sin(\xi x)}, \tag{30$_2$}$$
$$Q_1(x) = \frac{Q(1)\cos x - \gamma Q(\xi)\cos(\xi x)}{\cos x - \gamma\xi\cos(\xi x)}. \tag{30$_3$}$$

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REZIUMĖ

Šturmo ir Liuvilio uždavinio su Robino ir dvitaške kraštinėmis sąlygomis asimptotinė analizė

A. Štikonas

Mes analizuojame pradinį uždavinį ir gauname jo sprendinio asimptotinius skleidinius. Mes tiriame Sturmo ir Liuvilio uždavinį su su viena klasikine Robino tipo kraštine sąlyga ir kita dvitaške nelokalia kraštine sąlyga. Galiausiai gauname tikrinių reikšmių ir tikrinių funkcijų asimptotinius skleidinius.

Raktiniai žodžiai: Šturmo ir Liuvilio uždavinys; Robino sąlyga, dvitaškės nelokaliosios sąlygos, tikrinių reikšmių ir tikrinių funkcijų asimptotika