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On approximation of stochastic integrals with respect to a fractional Brownian motion

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Introduction

In the past few years, a fractional Brownian motion has been the subject of numerous investigations. The fBm B^H, $H > 1/2$, has the following kernel representation with respect to the standard Brownian motion:

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s,$$

where K_H is the deterministic kernel

$$K_H(t,s) = c_H\Big(H - \frac{1}{2}\Big)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}}\,\mathrm{d}u$$

with the normalizing constant c_H.

Let $\varkappa^n = \{t_k^n\colon\ 0 \leqslant k \leqslant n\}$ be the sequence of partitions of the interval $[0,1]$ with $t_k^n = k/n$. Define

$$A_t^n = \sum_{k=1}^{[nt]} n\int_{t_{k-1}^n}^{t_k^n} K_H\Big(\frac{[nt]}{n}, u\Big)\,\mathrm{d}u \cdot \frac{\xi_k^n}{\sqrt{n}},$$

where $[nt]$ denotes the integer part of nt, $\{\xi_k^n\}$ are i.i.d. random variables with $\mathbf{E}\xi_1^n = 0$ and $\mathbf{D}\xi_1^n = 1$. Sottinen [4] has proved that $A^n \xrightarrow{\mathcal{D}} B^H$ as $n \to \infty$.

Denote by $D([0,1],\mathbb{R}^d)$ the space of all mappings $x\colon\ [0,1] \to \mathbb{R}^d$ that are right-continuous and admit left-hand limits equipped with the Skorokhod $\mathcal{J}_1$ topology.

The aim of this note is to find conditions under which the convergence

$$\Big(X^n, A^n, \int_0^{\cdot} X_{s-}^n\,\mathrm{d}A_s^n\Big) \xrightarrow{\mathcal{D}} \Big(X, B^H, \int_0^{\cdot} X_s\,\mathrm{d}B_s^H\Big)$$

holds, where $\{X^n\}$ is a sequence of càdlàg processes such that

$$(X^n, A^n) \xrightarrow{\mathcal{D}} (X, B^H) \quad \text{in } D([0,1],\mathbb{R}^2).$$

Denote by $C^\alpha(\mathbb{R})$ the space of all continuous functions $g\colon \mathbb{R}\to\mathbb{R}$ such that

$$\sup_x |g(x)| + \sup_{x\neq y}\frac{|g(x)-g(y)|}{|x-y|^\alpha} < \infty.$$

THEOREM 1. *Let $0<\alpha<1$, $q>2$, and $1<p<2$ be such that $\frac{\alpha}{q}+\frac{1}{p}>1$. Let $\{X^n\}$ be a sequence of càdlàg processes with q-bounded variation, and let $f\in C^\alpha(\mathbb{R})$. Assume that $\sup_n \mathbf{E}v_q(X^n;[0,1])<+\infty$. If*

$$(X^n, A^n)\xrightarrow{\mathcal{D}}(X, B^H) \quad \text{in } D(\mathbb{R}_+,\mathbb{R}^2), \tag{1}$$

where X is a continuous process, then

$$\left(X^n, A^n, \int_0^\cdot f(X^n_{s-})\,\mathrm{d}A^n_s\right)\xrightarrow{\mathcal{D}}\left(X, B^H, \int_0^\cdot f(X_s)\,\mathrm{d}B^H_s\right) \quad \text{in } D([0,1],\mathbb{R}^3).$$

1. Basic notions and auxiliary results

The p-variation, $0<p<\infty$, of a real-valued function f on $[a,b]$ is defined as

$$v_p(f;[a,b]) = \sup_{\varkappa}\sum_{k=1}^n \bigl|f(x_k)-f(x_{k-1})\bigr|^p,$$

where the supremum is taken over all subdivisions $\varkappa=\{x_i\colon i=0,\dots,n\}$ of $[a,b]$ such that $a=x_0<x_1<\cdots<x_n=b$. Denote by $\mathcal{W}_p([a,b])$ the class of all functions on $[a,b]$ with bounded p-variation.

Let $f\in\mathcal{W}_q([a,b])$ and $h\in\mathcal{W}_p([a,b])$ with $0<p<\infty$, $q>0$, $1/p+1/q>1$. Then the integral $\int_a^b f\,\mathrm{d}h$ exists as the refinement Riemann–Stieltjes (RRS) integral if f and h have no common discontinuities on the same side at the same point. In particular, this necessary condition is satisfied if f is left-continuous and h is right-continuous or vice versa. If the integral exists, the Love–Young inequality

$$\left|\int_a^b f\,\mathrm{d}h - f(y)\bigl[h(b)-h(a)\bigr]\right| \leqslant C_{p,q}V_q(f;[a,b])V_p(h;[a,b]) \tag{2}$$

holds for all $y\in[a,b]$, where $C_{p,q}=\zeta(p^{-1}+q^{-1})$ and $\zeta(s)=\sum_{n\geqslant 1}n^{-s}$.

Let $f\in\mathcal{W}_p([a,b])$ and $p_1>p>0$. Then

$$V_{p_1}\bigl(f;[a,b]\bigr)\leqslant \operatorname{Osc}(f;[a,b])^{(p_1-p)/p_1}V_p^{p/p_1}\bigl(f;[a,b]\bigr), \tag{3}$$

where $\operatorname{Osc}(f;[a,b])=\sup\{|f(x)-f(y)|\colon x,y\in[a,b]\}$.

2. Proof of the theorem

Let $A_1,\dots,A_n$, $B_1,\dots,B_n$ be random variables such that $\mathbf{E}|A_i|^p$, $\mathbf{E}|B_i|^q<+\infty$, $i=1,\dots,n$. Let $S_{p,q}$ be the largest value of the products

$$\left(\sum_{k=1}^m \mathbf{E}|\overline{A}_k|^p\right)^{1/p}\left(\sum_{k=1}^m \mathbf{E}|\overline{B}_k|^q\right)^{1/q}$$

for which $\overline{A}_k = A_{i_k+1} + \ldots + A_{i_{k+1}}$ and $\overline{B}_k = B_{i_k+1} + \ldots + B_{i_{k+1}}$, $1 = i_1 < \ldots < i_k < \ldots < i_{m+1} = m$, $m \leqslant n$, are the corresponding sums of successive random variables A_i and B_i, respectively. A version of this lemma was proved in [2].

LEMMA 2. *Assume that* $\frac{1}{p} + \frac{1}{q} > 1$. *Then*
(i) *there exists an index* k $(1 \leqslant k \leqslant n)$ *such that*

$$\mathbf{E}|A_k B_k| \leqslant \left(\frac{1}{n}\sum_{k=1}^{m}\mathbf{E}|A_k|^p\right)^{1/p}\left(\frac{1}{n}\sum_{k=1}^{m}\mathbf{E}|B_k|^q\right)^{1/q},$$

(ii)

$$\mathbf{E}\left|-\sum_{1\leqslant i<j\leqslant k} A_i B_j + \sum_{k<i\leqslant j\leqslant n} A_i B_j\right| \leqslant C_{p,q} S_{p,q},$$

where $C_{p,q} = \zeta(p^{-1} + q^{-1})$.

The prove of this lemma is the same as that of the Love–Young inequality in [5], [3].

Proof of Theorem 1. Sottinen [4] has proved that, for all $s < t$,

$$\mathbf{E}|A^n(t) - A^n(s)|^2 \leqslant \left(\frac{[nt]}{n} - \frac{[ns]}{n}\right)^{2H} \quad \text{as } n \to \infty. \tag{4}$$

We show that

$$\sum_{k=1}^{[n\cdot]} f(X^n(t_{k-1}^n))\big(A^n(t_k^n) - A^n(t_{k-1}^n)\big) \xrightarrow{\mathcal{D}} \int_0^{\cdot} f(X(s))\,\mathrm{d}B^H(s), \quad n \to \infty. \tag{5}$$

Let us fix a partition of the interval [0, 1]. Denote it by $\varkappa^m$. By (1) we get

$$\big(X^n(t_0^m), A^n(t_0^m), \ldots, X^n(t_m^m), A^n(t_m^m)\big) \xrightarrow{\mathrm{d}} \big(X(t_0^m), B^H(t_0^m), \ldots, X(t_m^m), B^H(t_m^m)\big)$$

as $n \to \infty$. Thus,

$$\sum_{k=1}^{m} f(X^n(t_{k-1}^m))\big(A^n(t_k^m) - A^m(t_{k-1}^m)\big) \xrightarrow{\mathrm{d}} \sum_{k=1}^{m} f(X(t_{k-1}^m))\big(B^H(t_k^m) - B^H(t_{k-1}^m)\big) \quad \text{as } n \to \infty. \tag{6}$$

Assume that $m \leqslant n$ and set $\varkappa = \varkappa^m \cup \varkappa^n = \{t_k\colon t_k \in \varkappa^m \cup \varkappa^n\}$, $r(t) = \max\{k\colon t_k \leqslant t, t_k \in \varkappa\}$. Then

$$\left|\sum_{k=1}^{[mt]} f(X^n(t_{k-1}^m))\big(A^n(t_k^m) - A^n(t_{k-1}^m)\big) - \sum_{j=1}^{[nt]} f(X^n(t_{j-1}^n))\big(A^n(t_j^n) - A^n(t_{j-1}^n)\big)\right|$$

$$\leqslant \left| \sum_{k=1}^{[mt]} f(X^n(t_{k-1}^m))\big(A^n(t_k^m) - A^n(t_{k-1}^m)\big) - \sum_{j=1}^{r(t)} f(X^n(t_{j-1}))\big(A^n(t_j) - A^n(t_{j-1})\big) \right|$$

$$+ \left| \sum_{j=1}^{r(t)} f(X^n(t_{j-1}))\big(A^n(t_j) - A^n(t_{j-1})\big) - \sum_{k=1}^{[nt]} f(X^n(t_{k-1}^n))\big(A^n(t_k^n) - A^n(t_{k-1}^n)\big) \right|$$

$$= I_1^{m,n}(t) + I_2^{n,n}(t).$$

We estimate only $I_1^{m,n}$, since the proof of $I_2^{n,n}$ is the same.

Note that

$$I_1^{m.n}(t) = \left| \sum_{k=1}^{[mt]} \sum_{j \in \Delta_k^m} \big(f(X^n(t_{k-1}^m)) - f(X^n(t_{j-1}))\big)\big(A^n(t_j) - A^n(t_{j-1})\big) \right|$$

$$= \left| -\sum_{k=1}^{[mt]} \sum_{\substack{i<j \\ i,j \in \Delta_k^m}} \big(f(X^n(t_i)) - f(X^n(t_{i-1}))\big)\big(A^n(t_j) - A^n(t_{j-1})\big) \right|,$$

where $\Delta_k^m = \{j\colon\ t_j \in \varkappa, t_{k-1}^m < t_j \leqslant t_k^m\}$. Thus, by Lemma 2 we have

$$\mathbf{E} \sup_{t \leqslant 1} I_1^{m,n}(t) \leqslant \sum_{k=1}^{m} \mathbf{E} \left| - \sum_{\substack{i<j \\ i,j \in \Delta_k^m}} \big(f(X^n(t_i)) - f(X^n(t_{i-1}))\big)\big(A^n(t_j) - A^n(t_{j-1})\big) \right|$$

$$\leqslant C_{p,q/\alpha} \sum_{k=1}^{m} S_{p,q/\alpha}^{m,n,k},$$

where $S_{p,q/\alpha}^{m,n,k}$ is the largest value of the products

$$\left(\sum_{j=1}^{r} \mathbf{E}|f(X^n(s_j)) - f(X^n(s_{j-1}))|^{q/\alpha} \right)^{\alpha/q} \left(\sum_{j=1}^{r} \mathbf{E}|A^n(s_j) - A^n(s_{j-1})|^p \right)^{1/p} \tag{7}$$

for $t_{k-1}^m = s_0 \leqslant \ldots \leqslant s_r = t_k^m$, $s_j \in \varkappa$, $r \in \mathbb{N}$. Further, by (4), for $p > 1/H$, we have

$$\left(\sum_{j=1}^{r} \mathbf{E}|f(X^n(s_j)) - f(X^n(s_{j-1}))|^{q/\alpha} \right)^{\alpha/q} \left(\sum_{j=1}^{r} \mathbf{E}|A^n(s_j) - A^n(s_{j-1})|^p \right)^{1/p}$$

$$\leqslant |f|_\alpha \big(\mathbf{E} v_q(X^n; [t_{k-1}^m, t_k^m])\big)^{\alpha/q} \big(t_k^m - t_{k-1}^m\big)^H$$

and by the Hölder inequality

$$\sum_{k=1}^{m} S_{p,q/\alpha}^{m,n,k} \leqslant |f|_\alpha \left(\sum_{k=1}^{m} \mathbf{E} v_q(X^n; [t_{k-1}^m, t_k^m]) \right)^{\alpha/q} \left(\sum_{k=1}^{m} \big(t_k^m - t_{k-1}^m\big)^{pH} \right)^{1/p}$$

$$\leqslant |f|_\alpha \big(\mathbf{E} v_q(X^n;[0,1])\big)^{1/q} \max_{1\leqslant k\leqslant m} \big|t_k^m - t_{k-1}^m\big|^{H-1/p}.$$

Thus,

$$\mathbf{E}\sup_{t\leqslant 1} I_1^{m,n}(t) + \mathbf{E}\sup_{t\leqslant 1} I_2^{n,n}(t) \to 0 \quad \text{as } m,n\to\infty. \tag{8}$$

Denote

$$\widehat{X}^m(t) := X(t_k^m) \quad \text{for } t\in[t_{k-1}^m, t_k^m),\ 1\leqslant k\leqslant m-1,\ \text{or } t\in[t_{m-1}^m, t_m^m],$$

and

$$H^m(X) = \sup_{0\leqslant t\leqslant 1} \big|X(t) - \widehat{X}^m(t)\big|.$$

Let $q_1 > q$ and $\frac{\alpha}{q_1} + \frac{1}{p} > 1$. By inequality (3) we get

$$\begin{aligned}
&\sup_{t\leqslant 1}\left|\sum_{k=1}^{[mt]} f(X(t_{k-1}^m))\big(B^H(t_k^m) - B^H(t_{k-1}^m)\big) - \int_0^t f(X_s)\,\mathrm{d}B_s^H\right| \\
&\quad \leqslant \sup_{t\leqslant 1}\left|\sum_{k=1}^{[mt]} f(X(t_{k-1}^m))\big(B^H(t_k^m) - B^H(t_{k-1}^m)\big) - \int_0^{[mt]/m} f(X_s)\,\mathrm{d}B_s^H\right| \\
&\qquad + \sup_{t\leqslant 1}|f(X_t)| H^m(B^H) \\
&\quad \leqslant 4C_{p,q_1/\alpha}|f|_\alpha \big(H^m(X)\big)^{\alpha(q_1-q)/q_1} V_q^{\alpha q/q_1}\big(X;[0,1]\big) V_p\big(B^H;[0,1]\big) \\
&\qquad + |f|_\alpha V_q^\alpha\big(X;[0,1]\big) H^m(B^H).
\end{aligned} \tag{9}$$

Now by Lemma 4.2 of [1] and estimates (6), (8), and (9) we get (5).

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REZIUMĖ

K. Kubilius. Atsitiktinių integralų trupmeninio Brauno judesio atžvilgiu aproksimacijos klausimu

Nustatytos sąlygos, kada stohastinis integralas pasirinktos trupmeninio Brauno judesio aproksimacijos atžvilgiu silpnai konverguoja į stohastinį integralą trupmeninio Brauno judesio atžvilgiu.