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Weighted empirical FCLT for weakly dependent processes

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Abstract. This paper provides a weak convergence theorem for weighted empirical processes of a stationary sequence under a new weak dependence condition introduced by P. Doukhan and S. Louhichi.

Keywords: weak convergence, empirical process, weak dependence, Brownian Bridge.

1. Introduction

Let $(U_n)_{n\in Z}$ be a stationary sequence of random variables with common distribution function F. Assume without loss of generality that F is a uniform law on $[0, 1]$. Then the empirical distribution function E_n of $U_1, \dots, U_n$ is defined by

$$E_n(x) = n^{-1}\sum_{i=1}^{n} I(U_i \leqslant x), \quad -\infty < x < \infty,$$

where I is the usual indicator function. Define the empirical process

$$\alpha_n(t) = n^{1/2}\big(E_n(t) - t\big), \quad 0 < t < 1.$$

Let q be a positive weight function on $(0, 1)$. That is, $\inf_{\delta\leqslant t\leqslant 1-\delta} q(t) > 0$ for all $0 < \delta < 1/2$. And consider the following convergence result in the Skorohod space $D[0, 1]$ when the sample size converges to infinity

$$\frac{1}{q}\alpha_n \to^{D} \frac{1}{q}B^*. \tag{1}$$

Where B^* is the dependent analogue of a Brownian Bridge, that is B^* denotes the centered Gaussian process specified by $B^*(0) = B^*(1) = 1$ and

$$EB^*(x)B^*(y) = \sum_{k=-\infty}^{\infty} Cov\big(I(U_0 \leqslant x)I(U_{|k|} \leqslant y)\big). \tag{2}$$

Note that for independent sequences $\{U_n\}$ B^* coincides with the standard Brownian Bridge. For other important properties of B^* we refer to Shao, Yu [5]. The following theorem is the well known result from Chibisov (1964) and O'Reilly (1974).

THEOREM 1.1. *Let $\{U_i, i \in Z\}$ be i.i.d. r.v's. Assume that q is a positive and continuous weight function on $(0,1)$, and is nondecreasing in a neighborhood of 0 and non increasing in a neighborhood of 1. Then*

$$I(q,\lambda) := \int_0^1 \frac{1}{t(1-t)} \exp\Big(-\frac{\lambda q^2(t)}{t(1-t)}\Big)\, \mathrm{d}t < \infty \tag{3}$$

for all $\lambda > 0$ if and only if (1) holds in $D[0,1]$ with the Skorohod J_1-topology.

Function q which satisfies (3) for all $\lambda > 0$ is called a Chibisov–O'Reilly weight function. The critical function in this case is $\sqrt{t(1-t)}$, which doesn't satisfy relation (3).

The case of dependent random variables is considered in the paper by Q.M. Shao, H. Yu [5]. They proved the weighted Empirical CLT (1) for mixing and associated processes. More general dependent processes, in the context of an empirical limit theorem, are provided in the paper by P. Doukhan and S. Louhichi [2].

This work provides the weighted Empirical CLT (1) when a sequence $(U_n)_{n\in Z}$ is weakly dependent. With the help of Rozenthal-type inequalities under weak dependence it is shown that weighted ECLT holds for the weight functions having the following form

$$q(t) \geqslant C\big(t(1-t)\big)^{\mu}\Big(\log\frac{1}{t(1-t)}\Big)^{\beta}, \quad t \in (0,1), \tag{4}$$

where $C > 0$, $\beta > 1/2$, $0 < \mu \leqslant 1/2$ and μ depends on the form of weak dependence.

2. Dependence

2.1. Weak Dependence

Let L_n^∞ be the set of real valued bounded measurable functions on the Euclidean space R^n and let $\|.\|_\infty$ be the norm on L_n^∞. Consider a function h: $R^n \to R$ where R^n is equipped with the l^1-norm (i.e., $\|(z_1,\dots,z_n)\|_1 = |z_1| + \dots + |z_n|$) and define the Lipschitz modulus $Lip(h) = \sup_{x\neq y} \frac{|h(x)-h(y)|}{\|x-y\|_1}$. Let $L = \{h \in \bigcup_{i=1}^\infty L_i^\infty\colon\ \|h\|_\infty \leqslant 1,\ Lip(h) < \infty\}$.

DEFINITION 2.1 (P. Doukhan and S. Louhichi [2]). A sequence $(X_n)_{n\in Z}$ of random variables is called (θ, L, ψ)-weakly dependent if there exists a sequence $\theta = (\theta_r)_{r\in N}$ decreasing to zero at infinity and a real valued function ψ with arguments $(h,k,u,v) \in L \times L \times N \times N$ such that for any u-tuple $(i_1,\dots,i_u)$ and any v-tuple $(j_1,\dots,j_v)$ with $i_1 \leqslant \dots \leqslant i_u < i_u + r \leqslant j_1 \leqslant \dots \leqslant j_v$ one has

$$\big|Cov\big(h(X_{i_1},\dots,X_{i_u}), k(X_{j_1},\dots,X_{j_v})\big)\big| \leqslant \psi(h,k,u,v)\theta_r \quad \forall r \in N,$$

and the functions h and k are defined respectively on R^u and R^v.

Examples of interest include $\psi(h,k,u,v) = c(u,v)\mu(Lip(h), Lip(k))$ for some locally bounded functions μ and c. For example if we take $\mu(x,y) = c(x,y) = xy$ then we get a weak dependence condition typical for associated processes. An interesting case is a so called s-weak dependence where function $\psi(h,k,u,v) = v\|h\|_\infty Lip(k)$.

A wide class of weakly dependent sequences describes the following definition.

DEFINITION 2.2 Let $(\xi_i)_{i\in Z}$ be a sequence of i.i.d. real valued r.v's and let a function $H\colon R^Z \to R$ be measurable. A sequence $(X_n)_{n\in Z}$ of r.v. is called a Bernoulli shift if it is defined by $X_n = H(\xi_{n-j}, j\in Z)$. A sequence $(X_n)_{n\in Z}$ of r.v. is called causal Bernoulli shift if $X_n = H(\xi_n, \xi_{n-1}, \dots, \xi_0, \xi_{-1}, \dots)$, $H\colon R^N \to R$.

For any integer $k > 0$, denote $\delta_k = \sup_{i\in Z} E|H(\xi_{i-j}, j\in Z) - H(\xi_{i-j} I_{|j|<k}, j\in Z)|$. The following lemma describes the dependence structure of Bernoulli shifts.

LEMMA 2.1 (P. Doukhan, S. Louhichi [2]). *Mean zero Bernoulli shifts are (θ, L, ψ)-weakly dependent with*

$$\psi(h,k,u,v) = 4\big(u\|k\|_\infty Lip(h) + v\|h\|_\infty Lip(k)\big), \quad \theta_r = \delta_{r/2}.$$

Under causality, this holds with $\theta_r = \delta_r$ and $\psi(h,k,u,v) = 2vLip(k)\|h\|_\infty$.

2.2. *ECLT under dependence*

The following theorem of Q.M. Shao, H. Yu ([5]) is essential for the proofs of this paper.

THEOREM 2.2. *Let $\{U_n, n\geqslant 1\}$ be a stationary sequence of uniform $[0,1]$ random variables. Assume that for all $0\leqslant s,t\leqslant 1$ and $n\geqslant 1$ we have the following conditions*

$$\text{a)}\ \ E\big|\alpha_n(t)-\alpha_n(s)\big|^p \leqslant C_1\big(|t-s|^{p_1} + n^{-p_2/2}|t-s|^{r_1}\big)$$

for some $C_1>0$, $p>2$, $p_1>1$, $0\leqslant r_1\leqslant 1$ and $p_2 > 1-r_1$,

$$\text{b)}\ \ E\big|\alpha_n(t)-\alpha_n(s)\big|^2 \leqslant C_2|t-s|^{r_2}$$

for some $C_2>0$ and $0<r_2\leqslant 1$,

$$\text{c)}\ \ \alpha_n \to^D B^*.$$

Then (1) *holds with q an arbitrary weight function satisfying* (4) *for some $C>0$ and $\beta > 1/2$ and*

$$\mu = \min\Big(\frac{p_1}{p}, \frac{r_1+p_2}{p+p_2}, \frac{r_2}{2}\Big). \tag{5}$$

Note that if a Chibisov–O'Reilly weight function has the following form

$$q(t) = \Big(t(1-t)\log\log\big(1/(t(1-t))\big)\Big)^{1/2} f(t)$$

then, necessarily, $f(t)\to\infty$ as $t\to 0$ or $t\to 1$. Thus the weight function q from (4) can be compared to a Chibisov–O'Reilly weight function by taking μ in (5) close to $1/2$ or exactly $1/2$.

3. Weighted ECLT under weak dependence

Let's assume that $\{U_n, n \geqslant 1\}$ is a stationary sequence of uniform [0, 1] random variables satisfying the following condition

$$\sup_{f\in\Phi}\left|Cov\left(\prod_{i=1}^{2} f(U_{t_i}), \prod_{i=3}^{4} f(U_{t_i})\right)\right| \leqslant \theta_r, \tag{6}$$

where $\Phi = \{x \to I(s < x \leqslant t),\ \ s, t \in [0, 1]\ \}$, $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4$ and $r = t_3 - t_2$.

Note that if condition (6) is satisfied and $\theta_r = O(r^{-\frac{5}{2}-\nu})$, $\nu > 0$, then the sequence of processes α_n is tight in the Skorohod space $D[0, 1]$(see [1]).

The idea of this section is a use of (θ, L, ψ)-weak dependence together with the condition (6). Indeed the first one implies the second under the assumption of concentration of marginal distributions(in our case marginal distribution is uniform).

LEMMA 3.1. *Let $\{U_n, n \geqslant 1\}$ be a stationary sequence such that* (6) *holds. Assume that $\theta_r = O(r^{-a})$, $a > 5/2$ then for some constant $C > 0$*

$$E\left|\alpha_n(t) - \alpha_n(s)\right|^2 \leqslant C|t - s|^{\frac{a-1}{a}}.$$

Proof. Define $Y_n = I(s < U_n \leqslant t) - (t - s)$, $S_n = \sum_{i=1}^{n} Y_i$. By lemma 3 in P. Doukhan, S. Louhichi [2] it follows

$$E|S_n|^2 \leqslant Cn\sum_{r=1}^{n}\min\left(r^{-a}, |t - s|\right) \leqslant Cn\left(\sum_{r\geqslant|t-s|^{-\frac{1}{a}}} r^{-a} + \sum_{r<|t-s|^{-\frac{1}{a}}} |t - s|\right).$$

We can approximate the first sum by the integral

$$\int_{|t-s|^{-\frac{1}{a}}}^{n-1} \frac{du}{u^a} \leqslant C|t - s|^{\frac{a-1}{a}}.$$

the second sum also satisfies the bound $\leqslant C|t - s|^{\frac{a-1}{a}}$.

THEOREM 3.1. *Let $\{U_n, n \geqslant 1\}$ be a stationary sequence of uniform* [0, 1] *random variables. In addition assume that $\{U_n, n \geqslant 1\}$ is (θ, L, ψ_1)-weakly dependent, with*

$$\theta_r = O(r^{-5-\nu}), \quad \psi_1(h, k, u, v) = \left(Lip(h) \vee Lip(k)\right)(u + v), \quad \nu > 0,$$

then (1) *holds with an arbitrary weight function q satisfying* (4) *for some $C > 0$, $\beta > 1/2$ and $\mu = 1/2 - 1/(10 + 2\nu)$.*

Proof. Using Rosenthal type inequalities under weak dependence obtained by P. Doukhan, S. Louhichi [2] it follows that for $a = 5 + \nu$

$$E\left|\alpha_n(t) - \alpha_n(s)\right|^4 \leqslant C\left(\left(\sum_{r=0}^{n-1} r^{-a} \wedge |t - s|\right)^2 + \left(\frac{1}{n}\sum_{r=0}^{n-1}(r + 1)^2\theta_r\right)\right)$$

$$\leqslant C\Big(\Big(\sum_{r\geqslant|t-s|^{-\frac{1}{a}}} r^{-a}\Big)^2+\Big(\sum_{r<|t-s|^{-\frac{1}{a}}} |t-s|\Big)^2+n^{2-a}\Big)$$

$$\leqslant C\big(|t-s|^{\frac{2(a-1)}{a}}+n^{2-a}\big).$$

By Lemma 3.1 it follows that

$$E\big|\alpha_n(t)-\alpha_n(s)\big|^2\leqslant C|t-s|^{\frac{a-1}{a}}.$$

Note that from [2] it follows $\alpha_n \to^D B$. Now it is possible to use Theorem 2.2

$$p_1=\frac{2(a-1)}{a},\quad r_1=0,\quad p_2=2(a-2),\quad r_2=\frac{a-1}{a},\quad p=4.$$

Thus

$$\mu=\min\Big(\frac{p_1}{p},\frac{r_1+p_2}{p+p_2},\frac{r_2}{2}\Big)=\min\Big(\frac{a-2}{a},\frac{a-1}{2a}\Big)=\frac{a-1}{2a}=1/2-1/(10+2\nu).$$

References

1. P. Doukhan, *Models, Inequalities and Limit Theorems for Stationary sequences*, *Long Range Dependence*, Birkhaüser, (2001).
2. P. Doukhan, S. Louhichi, A new weak dependence condition and applications to moment inequalities, *Stochastic. Process. Appli.*, **84**, 313–342 (1999).
3. S. Louhichi, *Le Grade De Docteur En Sciences De L'Université Paris XI Orsay* (1998).
4. C. Prieur, *An Empirical Functional Central Limit Theorem for Weakly Dependent Sequences*, Université de Cergy-Pontoise (2000).
5. Q.M. Shao, H. Yu, Weak convergence for weighted empirical processes of dependent sequences, *Ann. Probab.*, **24**(4), 2098–2127 (1996).

REZIUMĖ

M. Juodis. Empirinė centrinė ribinė teorema su svoriais silpnai priklausomiems procesams

Darbe įrodoma centrinė ribinė teorema empiriniams procesams silpnai priklausomiems dydžiams. Nagrinėjamas empirinis procesas normuotas standartinėmis svorio funkcijomis.