

# Some remarks on selfnormalization for a simple spatial autoregressive model

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**Abstract.** In the paper I continue investigations on the self-normalization of simple autoregressive field  $X_{t,s} = aX_{t-1,s} + bX_{t,s-1} + \varepsilon_{t,s}$  started in [5]. And extend previous results when the variance of the innovations of the process above are not finite.

*Keywords:* autoregressive models, self-normalization, random fields.

## 1. Introduction and formulation of results

This paper continues the investigations of the self-normalization for the autoregressive fields on the plane which were started in [5]. In this paper the results for self-normalization for  $AR(1)$  process obtained by Juodis and Račkauskas [4] were generalized for the autoregressive field on the plane. However the results presented in [5] were proved under assumption of the the existence of the second moment for the innovations of the autoregressive field we are working with. Here we give a slight extension of this result and prove that sufficient condition for our results to hold is that innovations belong to the domain of attraction of the normal law.

We consider one of the most simple autoregressive fields

$$X_{t,s} = aX_{t-1,s} + bX_{t,s-1} + \varepsilon_{t,s}. \quad (1)$$

We suppose that that  $\varepsilon_{t,s}, (t, s) \in \mathbf{Z}^2$  are i.i.d. random variables with  $\varepsilon_{t,s} \in DAN$ . DAN here stands for domain of attraction of the normal law. We assume as well that  $|a| + |b| < 1$ . If the second moment for the innovations is finite this condition guarantees the existence of a stationary solution for (1). However this is not the case and we will have to investigate process (1) with initial conditions  $X_{0,s} = 0, X_{t,0} = 0, (t, s) \in \mathbf{Z}^2, t \geq 0, s \geq 0$ . In this case the process would have the form:

$$X_{t,s} = \sum_{k=0}^{t+s-1} \sum_{j=\max\{0, k-s+1\}}^{\min\{k, t-1\}} \binom{k}{j} a^j b^{k-j} \varepsilon_{t-j, s-k+j}, \quad t > 0, s > 0. \quad (2)$$

Let's define

$$\gamma_{(-1,1)} = \sum_{r=1}^{\infty} \sum_{s=1}^{r-1} \binom{r}{s} \binom{r}{s+1} a^{2s+1} b^{2r-2s-1}.$$

Set

$$\sigma^2(a, b) = (1 - a^2 - b^2)(1 - a - b)^{-2}(1 + 2ab\gamma_{(-1,1)})^{-1}.$$

Finally, let us denote  $\bar{M}_n = (M_{n,1}, M_{n,2})$  and

$$D_n = \{(t, s) : (t, s) \in \mathbf{Z}^2 \quad 1 \leq t \leq M_{n,1}, 1 \leq s \leq M_{n,2}\}.$$

(In the sequel, where there will be no confusion we suppress index  $n$  in some notations.) Now our first result can be formulated as follows.

**THEOREM 1.** *If  $X_{t,s}$  is defined by (2), random variables  $\varepsilon_{t,s}$  are i.i.d.,  $E\varepsilon_{t,s} = 0$   $\varepsilon_{t,s} \in DAN$ , and  $\min\{M_1, M_2\} \rightarrow \infty$  as  $n \rightarrow \infty$ , then:*

$$\frac{\sum_{(t,s) \in D_n} X_{t,s}}{(\sum_{(t,s) \in D_n} X_{t,s}^2)^{1/2}} \xrightarrow{\mathcal{D}} N(0, \sigma^2(a, b)). \tag{3}$$

To explain the meaning of notation  $\gamma_{(-1,1)}$  we revisit the case where innovations  $\varepsilon_{t,s}$  have finite variances. In this case for integers  $l_1, l_2$  we denote  $\gamma_{(l_1, l_2)} = EX_{t,s}X_{t+l_1, s+l_2}$ , where  $X_{t,s}$  is from (1).

In the paper [2] (see also[1] ) it was shown that

$$\gamma_{(-1,1)} := \begin{cases} \frac{(1-a^2-b^2)\gamma_{(0,0)}^2-1}{2ab}, & \text{if } ab \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\gamma_{(0,0)} := ((1+a+b)(1+a-b)(1-a+b)(1-a-b))^{-1/2}.$$

It is necessary to note that even in the case of infinite second moments for innovations, the limit distribution is still the same.

We will also reformulate the Theorem 2 in [5] with the new conditions for the innovations. Assume that  $1 \leq m_1 \leq M_1, 1 \leq m_2 \leq M_2$  are integers (dependent on  $n$ , too) and such that  $I := M_1m_1^{-1}$  and  $J := M_2m_2^{-1}$  are integers, we set  $m_{1,i} := m_1(i - 1), m_{2,j} := m_2(j - 1)$  and

$$D_{i,j} = \{(t, s) : m_{1,i} + 1 \leq t \leq m_{1,i+1}, m_{2,j} + 1 \leq t \leq m_{2,j+1}\},$$

$$1 \leq i \leq I, \quad 1 \leq j \leq J.$$

If we define

$$Y_{i,j} = \sum_{(t,s) \in D_{i,j}} X_{t,s},$$

then, since  $D_n = \cup_{i=1}^I \cup_{j=1}^J D_{i,j}$ , clearly we have  $\sum_{(t,s) \in D_n} X_{t,s} = \sum_{i=1}^I \sum_{j=1}^J Y_{i,j}$ .

**THEOREM 2.** *If conditions of Theorem 1 are satisfied and additionally*

$$\min(m_1, m_2) \rightarrow \infty; \quad \frac{m_1m_2}{M_1M_2} \rightarrow 0,$$

as  $n \rightarrow \infty$ , then

$$\frac{\sum_{(t,s) \in D_n} X_{t,s}}{(\sum_{i=1}^I \sum_{j=1}^J Y_{i,j}^2)^{1/2}} \xrightarrow{\mathcal{D}} N(0, 1).$$

### 2. Proofs

Due to the short format of the paper we find it impossible to place all of the proof here so we will only quote what we need to change in [5] in order to prove theorems formulated above and present only the most interesting part in more detail putting the simpler parts aside.

We reintroduce notations used in [5]:

$$\chi_n^2 := \sum_{t=1}^{M_{n,1}} \sum_{s=1}^{M_{n,2}} \epsilon_{t,s}^2, \quad R_n := |D_n| = M_{n,1} M_{n,2}.$$

In order to verify **Theorem 1** we need to prove:

$$\left( b \sum_{t=1}^{M_1} X_{t,M_2} + a \sum_{s=1}^{M_2} X_{M_1,s} \right) \chi_n^{-1} \xrightarrow{P} 0, \tag{4}$$

$$\left( b \sum_{t=1}^{M_1} X_{t,M_2}^2 + a \sum_{s=1}^{M_2} X_{M_1,s}^2 \right) \chi_n^{-1} \xrightarrow{P} 0, \tag{5}$$

$$\left( 2a \sum_{(t,s) \in D_n} X_{t-1,s} \epsilon_{t,s} + 2b \sum_{(t,s) \in D_n} X_{t,s-1} \epsilon_{t,s} \right) \chi_n^{-2} \xrightarrow{P} 0, \tag{6}$$

$$2ab \left( \sum_{(t,s) \in D_n} X_{t-1,s} X_{t,s-1} \right) \chi_n^{-2} \xrightarrow{P} 2ab\gamma_{(-1,1)}. \tag{7}$$

(4) is equivalent to proving  $R_n^{(1)} \chi_n^{-1} \xrightarrow{P} 0$  in [5], analogously (5) is equivalent to proving  $R_n^{(3)} \chi_n^{-1} \xrightarrow{P} 0$ , (6) to  $R_n^{(2)} \chi_n^{-2} \xrightarrow{P} 0$  and finally (7) is equivalent to proving that  $F_n^{(2)} \chi_n^{-2} \xrightarrow{P} 2ab\gamma_{(-1,1)}$  in [5].

To prove **Theorem 2** we would need to prove additionally:  $\sum_{i,j} \kappa_{i,j}^2 \chi_n^{-2} \xrightarrow{P} 0$ ,  $\sum_{i,j} \eta_{i,j}^2 \chi_n^{-2} \xrightarrow{P} 0$ . For the definitions we have to redirect you to [5].

Part (7) is most interesting and we will provide some details of its proof here.

We need to show that

$$2ab \left( \sum_{(t,s) \in D_n} X_{t-1,s} X_{t,s-1} \right) \chi_n^{-2} \xrightarrow{P} 2ab\gamma_{(-1,1)} \tag{8}$$

for the process defined by (2).

Let's denote:

$$X_{t-1,s} X_{t,s-1} = Z_{t,s}^{(1)} + Z_{t,s}^{(2)}.$$

Using (2) we define  $Z_{t,s}^{(1)}$  and  $Z_{t,s}^{(2)}$  like this:

$$\begin{aligned} Z_{t,s}^{(1)} &= \sum_{k=1}^{t+s-2} \sum_{j=\max\{0, k-s+1\}}^{\min\{t-2, k-1\}} \binom{k}{j} \binom{k}{j+1} a^{2j+1} b^{2k-2j-1} \varepsilon_{t-j, s-k+j}^2, \\ Z_{t,s}^{(2)} &= \sum_{\substack{k, k_1=0 \\ t-1-j \neq t-j_1 \quad \text{or} \quad s-k+j \neq s-1-k_1+j_1}}^{t+s-2} \sum_{j_1=\max\{0, k_1-s+1\}}^{\min\{t-2, k_1\}} \sum_{j=\max\{0, k-s+2\}}^{\min\{t-1, k\}} \binom{k}{j} \binom{k_1}{j_1} a^{j+j_1} b^{k+k_1-j-j_1} \\ &\quad \times \varepsilon_{t-1-j, s-k+j} \varepsilon_{t-j_1, s-1-k_1+j_1}. \end{aligned}$$

The sum  $Z_{t,s}^{(1)}$  contains all the members of the product  $X_{t-1,s} X_{t,s-1}$ , where  $t-1-j = t-j_1$  and  $s-k+j = s-1+k_1+j_1$ , sum  $Z_{t,s}^{(2)}$  has the rest.

This means that expectations  $E|\varepsilon_{t',s'} \varepsilon_{t'',s''} \chi_n^{-2}|$  in sums  $Z_{t,s}^{(1)} \chi_n^{-2}$  and  $Z_{t,s}^{(2)} \chi_n^{-2}$  can be estimated using Lemmas 2 and 3 from [5] (which were originally proved in [3])

$$E|\varepsilon_{t,s}^2 \chi_n^{-2}| \leq C R_n^{-1} \quad (9)$$

for the summands in  $Z_{t,s}^{(1)}$  and

$$E|\varepsilon_{t-1-j, s-k+j} \varepsilon_{t-j_1, s-1-k_1+j_1} \chi_n^{-2}| \leq C R_n^{-2} \quad (10)$$

for the summands in  $Z_{t,s}^{(2)}$ .  $C$  here and anywhere else in the text stands for a constant.

$$\sum_{(t,s) \in D_n} Z_{t,s}^{(2)} \chi_n^{-2} \xrightarrow{P} 0$$

follows then from the usual Chebyshev inequality and the fact:

$$\sum_{k, k_1=0}^{t+s-2} \sum_{j=0}^k \sum_{j_1=0}^{k_1} \binom{k}{j} \binom{k_1}{j_1} |a|^{j+j_1} |b|^{k+k_1-j-j_1} \leq (1-d)^{-2}.$$

Here  $d = |a| + |b|$ .

The proof that  $\sum_{(t,s) \in D_n} Z_{t,s}^{(1)} \chi_n^{-2} \xrightarrow{P} \gamma_{(-1,1)}$  is just marginally more complex. As we know

$$\gamma_{(-1,1)} = \sum_{r=1}^{\infty} \sum_{s=0}^{r-1} \binom{r}{s} \binom{r}{s+1} a^{2s+1} b^{2(r-s)-1}.$$

Denote  $M_{1,n} - t = h_{1,t}$  and  $M_{2,n} - s = h_{2,s}$ . If we regroup the coefficients by  $\epsilon_{t,s}$  in sum  $\sum_{(t,s) \in D_n} Z_{t,s}^{(1)}$  we would get:

$$\sum_{(t,s) \in D_n} Z_{t,s}^{(1)} = \sum_{(t,s) \in D_n} \left( \sum_{k=1}^{h_{1,t}+h_{2,s}-1} \sum_{j=\max\{0, k-h_{2,s}+1\}}^{\min\{k-1, h_{1,t}-1\}} \binom{k}{j} \binom{k}{j+1} a^{2j+1} b^{2k-2j-1} \right) \epsilon_{t,s}.$$

Thus by each of the  $\epsilon_{t,s}$  in sum  $\sum_{(t,s) \in D_n} Z_{t,s}^{(1)} = \sum_{(t,s) \in D_n} \tau_{t,s} \epsilon_{t,s}$  stands a trimmed version of  $\gamma_{(-1,1)}$ . Here

$$\tau_{t,s} = \sum_{k=1}^{h_{1,t}+h_{2,s}-1} \sum_{j=\max\{0, k-h_{2,s}+1\}}^{\min\{k-1, h_{1,t}-1\}} \binom{k}{j} \binom{k}{j+1} a^{2j+1} b^{2k-2j-1} = \gamma_{(-1,1)} - R_{t,s}^{(n)}.$$

We split the remainder term  $R_{t,s}^{(n)}$  into 3 parts:

$$R_{t,s}^{(n)} = R_{t,s}^{(n,1)} + R_{t,s}^{(n,2)} + R_{t,s}^{(n,3)}.$$

Definitions are as follows:

$$\begin{aligned} R_{t,s}^{(n,1)} &= \sum_{k=h_{1,t}+h_{2,s}}^{\infty} \sum_{j=0}^{k-1} \binom{k}{j} \binom{k}{j+1} a^{2j+1} b^{2k-2j-1}, \\ R_{t,s}^{(n,2)} &= \sum_{k=1}^{h_{1,t}+h_{2,s}-1} \mathbf{1}_{k-h_{2,s}+1 > 0} \sum_{j=0}^{k-h_{2,s}} \binom{k}{j} \binom{k}{j+1} a^{2j+1} b^{2k-2j-1}, \\ R_{t,s}^{(n,3)} &= \sum_{k=1}^{h_{1,t}+h_{2,s}-1} \mathbf{1}_{k-h_{1,t} > 0} \sum_{j=h_{1,t}}^{k-1} \binom{k}{j} \binom{k}{j+1} a^{2j+1} b^{2k-2j-1}. \end{aligned}$$

The  $R_{t,s}^{(n,2)}$ , and  $R_{t,s}^{(n,3)}$  are similar so we will just show how to estimate the impact of one of them, the other one is dealt with in the same way.

The estimation is as follows:

$$\begin{aligned} |R_{t,s}^{(n,2)}| &= \left| \sum_{k=h_{2,s}+2}^{h_{1,t}+h_{2,s}-1} \sum_{j=0}^{k-h_{2,s}} \binom{k}{j} \binom{k}{j+1} a^{2j+1} b^{2k-2j-1} \right| \\ &\leq |a||b| \sum_{k=h_{2,s}}^{h_{1,t}+h_{2,s}-1} (k+1) \sum_{j=0}^{k+1-h_{2,s}} \left( \binom{k}{j} \binom{k}{j} a^j b^{k-j} \right)^2 \\ &\leq |a||b| \sum_{k=h_{2,s}}^{h_{1,t}+h_{2,s}-1} (k+1) d^{2k} \leq C h_{2,s} d^{2h_{2,s}}. \end{aligned}$$

Now we can easily see that Chebyshev theorem and estimate (9) proves:

$$\sum_{(t,s) \in D_n} R_{t,s}^{(n,2)} \epsilon_{t,s}^2 \chi_n^{-2} \xrightarrow{P} 0.$$

We would get the same result for  $R_{t,s}^{(n,3)}$  following the same steps.  
To prove that

$$\sum_{(t,s) \in D_n} R_{t,s}^{(n,1)} \epsilon_{t,s}^2 \chi_n^{-2} \xrightarrow{P} 0,$$

we use the same approach and similar estimate.

$$|R_{t,s}^{(n,1)}| \leq C(h_{1,t} + h_{2,s})d^{2(h_{1,t} + h_{2,s})}.$$

Thus the required result (8) is achieved. The proof of (6) follows from the combination of Chebyshev theorem and aforementioned lemmas 2 and 3 from [5]. However in this case estimation of the second moment of the quantity is necessary, but the framework is essentially the same. Proving (4) and (5) is even simpler, it can be done just by applying lemmas and using Newton binomial formula.

### References

1. S. Baran, G. Pap, M.C.A. van Zuijlen, Asymptotic inference for unit roots in spatial triangular autoregression, Department of Math., Radboud university Nijmegen, *Report No. 0506*, 1–49 (2005).
2. S. Basu, G.C. Reinsel, Properties of the spatial unilateral first order arma model, *Adv. Appl. Prob.*, **25**, 631–648 (1996).
3. E. Gine, F. Gotze, D.M. Mason, When is the student  $t$ -statistic asymptotically standard normal?, *Ann. Probab.*, **25**, 1514–1531 (1997).
4. M. Juodis, A. Rackauskas, A remark on self-normalization for dependent random variables, *Lith. Math. J.* (to appear).
5. V. Paulauskas, R. Zovė, A note on selfnormalization for a simple spatial autoregressive model, *Lith. Math. J.*, **47**(2), 220–233 (2007).

### REZIUMĖ

#### **R. Zovė. Keletas pastabų apie paprasto autoregresinio lauko autonormavimą**

Straipsnyje praplėstas paprasto autoregresinio lauko  $X_{t,s} = aX_{t-1,s} + bX_{t,s-1} + \epsilon_{t,s}$  autonormavimo tyrimas pradėtas [5]. Gauti rezultatai kuomet jau minėto lauko inovacijos neturi antro momento.