

Spectral problem for the mean field Hamiltonian

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Abstract. We consider the mean field Hamiltonian $\bar{\mathcal{H}}_V = \kappa \bar{\Delta}_V + \xi(\cdot)$ in $l^2(V)$, where $V = \{x\}$ is a finite set. Characteristic equations for eigenvalues and expressions for eigenfunctions of $\bar{\mathcal{H}}_V$ are obtained. Using this result, the spectral representation of the solution of the corresponding ("heat transition") differential equation is derived.

Keywords: mean field difference operator, eigenvalue problem, characteristic equation for eigenvalues, linear differential equations, spectral representation of solutions.

1. Introduction

Let $V = \{x\}$ be a finite set, and let N be the number of elements of V . The mean field (Curie-Weiss) model in V is given by the symmetric operator (N -square matrix) $\bar{\mathcal{H}}_V$, acting on functions (N -dimensional vectors) $\psi(\cdot): V \rightarrow \mathbb{R}$ according to the formula

$$\bar{\mathcal{H}}_V \psi(x) = \kappa \bar{\Delta}_V \psi + \xi(x) \psi(x), \quad x \in V, \quad (1)$$

where $\bar{\Delta}_V \psi = N^{-1} \sum_{x \in V} \psi(x)$, κ is a positive constant and the potential $\xi(\cdot) = \{\xi(x): x \in V\}$ consists of real scalars. In this paper, we obtain equations for eigenvalues and derive formulas for eigenfunctions of the Hamiltonian $\bar{\mathcal{H}}_V$ (Theorem 1). Theorem 1 is applied to obtain the spectral representation of the solution $u(t, x)$ to the ("heat transition") differential equation

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \frac{\kappa}{N} \sum_{y \in V} (u(t, y) - u(t, x)) + \xi(x) u(t, x), \\ u(0, x) = u_0(x), \quad t \geq 0, \quad x \in V. \end{cases} \quad (2)$$

(Theorem 6). The Feynman-Kac formula for $u(t, x)$ is also discussed (Theorem 5).

The Hamiltonian (1) is a simplified modification of the lattice Schrödinger operator $\mathcal{H}_V = \kappa \Delta_V + \xi(\cdot)$ in $l^2(V)$, $V \subset \mathbb{Z}^p$, with the lattice Laplacian Δ_V (cf. [3]).

In the case of independent identically distributed random variables $\xi(x)$, $x \in V$, with distribution function $F(s) = \mathbb{P}(\xi(x) \leq s)$, the spectral problem for the operator (1) and the asymptotic behavior (as $N \rightarrow \infty$ and $t \rightarrow \infty$) of the extreme eigenvalues and the solution $u(t, x)$ of equation (2) were discussed in [2] (Gaussian distributions), [4] (exponential distributions) and [1] (continuous distribution functions $F(\cdot)$). We note that, for a continuous $F(\cdot)$, the variables $\xi(x)$, $x \in V$, are all distinct with probability one. In this paper, we consider the general case of scalars $\xi(x)$, $x \in V$.

In Section 1, we study the eigenvalue problem for $\bar{\mathcal{H}}_V$. Section 2 is devoted to the discussion of representations of the function $u(\cdot, \cdot)$.

2. Spectral problem

We consider the spectral problem

$$\bar{\mathcal{H}}_V \psi(\cdot) = \lambda \psi(\cdot), \quad \lambda \in \mathbb{R}, \quad \psi(\cdot) = \{\psi(x) : x \in V\} \in \mathbb{R}^N, \quad (3)$$

with $\psi(\cdot) \neq 0$ normed by the condition $\sum_{x \in V} \psi^2(x) = 1$; here, remember, $N = |N|$.

THEOREM 1. *Let*

$$\xi_{1,V} \geq \xi_{2,V} \geq \dots \geq \xi_{N,V} \quad (4)$$

be the variational series of the scalars $\xi(x)$, $x \in V$, and assume that there are exactly $L \geq 1$ strict inequalities in (4):

$$\xi_{k_1,V} > \xi_{k_2,V} > \dots > \xi_{k_L,V},$$

where, without loss of generality, $k_1 = 1$. Set $\xi_{k_0,V} = \infty$.

Then the Hamiltonian $\bar{\mathcal{H}}_V$ has N real eigenvalues $\lambda_{1,V} \geq \lambda_{2,V} \geq \dots \geq \lambda_{N,V}$ which are specified as follows:

(i) for each $i = 1, 2, \dots, L$, there is one eigenvalue $\lambda_{k_i,V}$ in the (open) interval $(\xi_{k_{i-1},V}, \xi_{k_i,V})$ satisfying the equation

$$\sum_{x \in V} \frac{1}{\lambda - \xi(x)} = \frac{N}{\kappa} \quad (5)$$

and the corresponding (normed) eigenfunction has the form

$$\psi_{k_i}(x) = (\lambda_{k_i,V} - \xi(x))^{-1} \left(\sum_{y \in V} (\lambda_{k_i,V} - \xi(y))^{-2} \right)^{-1/2}, \quad x \in V; \quad (6)$$

(ii) for each $i = 1, 2, \dots, L$, if the multiplicity of $\xi_{k_i,V}$ in (4) is $m_i \geq 2$ (i.e., there is the maximal subset $\{x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)} \in V\}$ such that $\xi_{k_i,V} = \xi(x_1^{(i)}) = \xi(x_2^{(i)}) = \dots = \xi(x_{m_i}^{(i)})$), then $\lambda = \xi_{k_i,V}$ is an eigenvalue with multiplicity $m_i - 1$ and the set of (normed) eigenfunctions associated with λ can be chosen as an orthonormal basis of the $(m_i - 1)$ -dimensional subspace $\{\psi(\cdot) \in \mathbb{R}^N : \psi(x_1^{(i)}) + \psi(x_2^{(i)}) + \dots + \psi(x_{m_i}^{(i)}) = 0 \text{ and } \psi(x) = 0 \text{ for each } x \in V \setminus \{x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)}\}\}$.

Theorem 1 is proved below.

Remark 2. With notation of part (ii) of Theorem 1, we have that $k_1 m_1 + k_2 m_2 + \dots + k_L m_L = N$ and $\bigcup_{i=1}^L \{x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)}\} = V$.

Remark 3. Since $\bar{\mathcal{H}}_V$ is a symmetric N -square matrix, from Theorem 1 we see that the eigenfunctions $\psi_1(\cdot), \psi_2(\cdot), \dots, \psi_N(\cdot)$ of $\bar{\mathcal{H}}_V$ form an orthonormal basis of the N -dimensional vector space \mathbb{R}^N .

Remark 4. From Theorem 1 we obtain that

$$\lambda_{1,V} > \xi_{1,V} \geq \lambda_{2,V} \geq \xi_{2,V} \geq \dots \geq \xi_{N-1,V} \geq \lambda_{N,V} \geq \xi_{N,V}.$$

In addition, if $\xi_{i-1,V} > \xi_{i,V}$, then $\xi_{i-1,V} > \lambda_{i,V} > \xi_{i,V}$; meanwhile, if $\xi_{i-1,V} = \xi_{i,V}$; then $\lambda_{i,V} = \xi_{i,V}$ ($i = 2, 3, \dots, N$).

Proof of Theorem 1. We rewrite (3) in the following form

$$(\lambda - \xi(x))\psi(x) = \bar{\Delta}_V \psi, \quad x \in V. \quad (7)$$

To prove part (i) of the theorem, assume that $\lambda \neq \xi(x)$ for each x . From (7) we obtain that

$$\sum_{x \in V} \psi(x) = N \bar{\Delta}_V \psi = \kappa \bar{\Delta}_V \psi \sum_{x \in V} \frac{1}{\lambda - \xi(x)},$$

i.e.,

$$\frac{N}{\kappa} \bar{\Delta}_V \psi = \bar{\Delta}_V \psi \sum_{x \in V} \frac{1}{\lambda - \xi(x)}. \quad (8)$$

Assume now that $\sum_{x \in V} \psi(x) = 0$. Since $\lambda \neq \xi(x)$ for each x , from (7) we see that $\psi(\cdot) = 0$, i.e., the eigenfunction $\psi(\cdot)$ associated with eigenvalue λ is necessarily zero. This contradicts the definition of the eigenfunction, therefore, $\bar{\Delta}_V \psi \neq 0$.

Since $\bar{\Delta}_V \psi \neq 0$, from (8) we obtain the characteristic equation (5) for eigenvalues. By $g(\lambda)$ we denote the left-hand side of (5). Note that, for each $i = 1, 2, \dots, L$, $g(\lambda) \rightarrow \pm\infty$ as $\lambda \rightarrow \xi_{k_i,V} \pm 0$ and $g(\lambda) \rightarrow 0$ as $\lambda \rightarrow \pm\infty$. Therefore, in $\mathbb{R} \setminus \{\xi_{k_1,V}, \xi_{k_2,V}, \dots, \xi_{k_L,V}\}$ equation (5) has exactly L eigenvalues $\lambda_{k_i,V} \in (\xi_{k_{i-1},V}, \xi_{k_i,V})$, and by (7) the corresponding (normed) eigenfunctions $\psi_{k_i}(\cdot)$ are defined by (6); $i = 1, 2, \dots, L$. Thus part (i) is proved.

To prove part (ii), let us consider the equation (7). Fix $i = 1, 2, \dots, L$ and the subset $\{x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)} \in V\}$ with $m_i \geq 2$ such that $\xi_{k_i,V} = \xi(x_1^{(i)}) = \xi(x_2^{(i)}) = \dots = \xi(x_{m_i}^{(i)})$. If $\lambda = \xi_{k_i,V}$, then (7) implies that $\bar{\Delta}_V \psi = 0$. Therefore, from (7) we have that $\psi(x) = 0$ for each $x \in V \setminus \{x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)}\}$. Thus $\psi(x_1^{(i)}) + \psi(x_2^{(i)}) + \dots + \psi(x_{m_i}^{(i)}) = 0$. Summarizing, we see that if $\xi_{k_i,V} = \xi(x_1^{(i)}) = \xi(x_2^{(i)}) = \dots = \xi(x_{m_i}^{(i)})$ with $m_i \geq 2$, then the eigenspace of the eigenvalue $\lambda = \xi_{k_i,V}$ is the $(m_i - 1)$ -dimensional subspace $\{\psi(\cdot) \in \mathbb{R}^N: \psi(x_1^{(i)}) + \psi(x_2^{(i)}) + \dots + \psi(x_{m_i}^{(i)}) = 0 \text{ and } \psi(x) = 0 \text{ for each } x \in V \setminus \{x_1^{(i)}, x_2^{(i)}, \dots, x_{m_i}^{(i)}\}\}$. I.e., part (ii) is proved.

3. Application to evolution systems

Let us consider the evolution system described by the equation (2). Recall that the operator $\bar{\Delta}_V \psi - \kappa \psi(\cdot)$ ($\psi(\cdot): V \rightarrow \mathbb{R}$) is a generator of the random walk $\bar{x} = \{\bar{x}_t: t \geq 0\}$ in V with continuous time which stays at any site during the time, distributed exponentially with parameter $\kappa > 0$ and then takes a jump to one of sites in V with probability

$1/N$ (a mean field random walk, a totally symmetric random walk). In other words, the local transition probabilities of the homogeneous random walk \bar{x} are given by the formula

$$\mathbb{P}(\bar{x}_{t+\Delta t} = y | \bar{x}_t = x) = \begin{cases} \frac{\kappa}{N} \Delta t + o(\Delta t), & \text{in the case of a jump,} \\ 1 - \kappa \Delta t + o(\Delta t), & \text{otherwise,} \end{cases} \quad (9)$$

as $\Delta t \rightarrow 0$, for all $x \in V$ and $y \in V$. The equation (2) is to describe the evolution of the system of noninteracting particles in V . Each particle moves according to the random walk \bar{x} (diffusion of the system). Additionally, each particle situated at $x \in V$ splits into two particles at the same x with probability $\max(\xi(x), 0)\Delta t + o(\Delta t)$ and disappears with probability $\max(-\xi(x), 0)\Delta t + o(\Delta t)$ during time interval $(t; t + \Delta t)$ (branching mechanism of the system). Then $u(t, x)$ is the mean number of particles at site x at time t (cf. [6]).

THEOREM 5 (Feynman-Kac formula). *Equation (2) has a unique nonnegative solution $u(\cdot, \cdot)$ represented as an integral over paths:*

$$u(t, x) = \mathbb{E}_x \left[\exp \left\{ \int_0^t \xi(\bar{x}_s) ds \right\} u_0(\bar{x}_t) \right], \quad t \geq 0, x \in V,$$

where the expectation \mathbb{E}_x is taken with respect to the mean field random walk \bar{x} which starts at $x \in V$.

Proof. Using a strong Markovian property and local transition probabilities (9) of the random walk \bar{x} , we have that

$$\begin{aligned} u(t + \Delta t, x) &= \mathbb{E}_x \left[\exp \left\{ \int_0^{t+\Delta t} \xi(\bar{x}_s) ds \right\} u_0(\bar{x}_{t+\Delta t}) \right] \\ &= \mathbb{E}_x \left[\exp \left\{ \int_0^{\Delta t} \xi(\bar{x}_s) ds \right\} \mathbb{E}_{\bar{x}_{\Delta t}} \left(\exp \left\{ \int_{\Delta t}^{t+\Delta t} \xi(\bar{x}_s) ds \right\} u_0(\bar{x}_{t+\Delta t}) \right) \right] \\ &= \mathbb{E}_x \left[\exp \left\{ \int_0^{\Delta t} \xi(\bar{x}_s) ds \right\} u(t, \bar{x}_{\Delta t}) \right] \\ &= e^{\Delta t \xi(x)} u(t, x) (1 - \kappa \Delta t + o(\Delta t)) + e^{O(\Delta t)} \sum_{y \in V} u(t, y) \left(\frac{\kappa}{N} \Delta t + o(\Delta t) \right) \\ &= u(t, x) + \frac{\kappa}{N} \sum_{y \in V} (u(t, y) - u(t, x)) \Delta t + \xi(x) u(t, x) \Delta t + o(\Delta t) \end{aligned}$$

as $\Delta t \rightarrow 0$. We take $u(t, x)$ to the left-hand side and divide both sides by Δt . Passing to the limit as $\Delta t \rightarrow 0$, we obtain the assertion of Theorem 5.

THEOREM 6 (Spectral representation). *The solution $u(\cdot, \cdot)$ of equation (2) is expanded over the eigenvalues $\lambda_{k,V}$ and eigenfunctions $\psi_k(\cdot)$ ($1 \leq k \leq N$) of the Hamil-*

tonian $\bar{\mathcal{H}}_V = \kappa \bar{\Delta}_V + \bar{\xi}(\cdot)$, viz.

$$u(t, x) = \sum_{k=1}^N \exp\{t\lambda_{k,V} - t\kappa\} \psi_k(x) (\psi_k, u_0)_V, \quad t \geq 0, x \in V$$

(cf. Theorem 1); here $(\psi_k, u_0)_V = \sum_{x \in V} \psi_k(x) u_0(x)$ is the inner product of $\psi_k(\cdot)$ and $u_0(\cdot)$.

Proof. Since the eigenfunctions $\psi_1(\cdot), \psi_2(\cdot), \dots, \psi_N(\cdot)$ of $\bar{\mathcal{H}}_V$ form an orthonormal basis of \mathbb{R}^N , the function $u(t, \cdot)$ is expanded over $\psi_k(\cdot)$, viz.

$$u(t, \cdot) = \sum_{k=1}^N a_k(t) \psi_k(\cdot)$$

for each $t \geq 0$. Substituting this into (2) and noting that the eigenfunctions $\psi_k(\cdot)$ are linearly independent, we obtain the equations for $a_k(\cdot)$:

$$\frac{da_k(t)}{dt} = (\lambda_{k,V} - \kappa) a_k(t), \quad t \geq 0;$$

here $1 \leq k \leq N$. This equation has the solution $a_k(t) = c_k \exp\{t\lambda_{k,V} - t\kappa\}$, where $c_k = (\psi_k, u_0)_V$ by calculating the inner product of $\psi_k(\cdot)$ and $u_0(\cdot) = \sum_{l=1}^N c_l \psi_l(\cdot)$. Summarizing, we obtain the assertion of the theorem.

For the theory of linear differential equations we refer to, e.g., [5].

References

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REZIUMĖ

A. Astrauskas. Vidurkinio lauko Hamiltoniano tikrinių reikšmių uždavinys

Darbe gautos vidurkinio lauko operatoriaus tikrinių reikšmių charakteringoji lygtis bei tikrinių funkcijų išraiškos. Šie rezultatai taikomi tiriant atitinkamos tiesinės diferencialinės lygties sprendinio sklaidimą tikrinėmis funkcijomis.