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Present value of firm asset in case of correlated defaults: a generalized structural approach of credit risk

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Abstract. This article investigates the present value of a firm's asset in the case of $n \geqslant 2$ correlated defaults. The structural approach of credit risk is developed in the case when default boundaries follow geometric Brownian motions. Correlated defaults are defined by the implied correlation of Brownian motions. The operational risk and the risk of financial market changes are allowed in this model. Also, the impact of implied correlation to the present value of firm's asset is shown numerically.

Keywords: correlated defaults, implied correlation, present value.

1. Introduction

In the context of credit risk of defaul table bonds portfolio, it is important to know the present value of firm's (i.e., debtor's) asset at maturity when the defaults are correlated. In this paper, we investigate the portfolio of $n \geqslant 2$ correlated defaultable bonds. We generalize some theorems for truncated mean of multidimensional firms' asset value process (i.e., multidimensional Brownian motion) concerning the evaluation of vulnerable call option (see [1]) in case when default boundaries follow geometric Brownian motion. We give its interpretation in the context not only of credit risk but also for additional operational risk and the risk of market change, i.e., of whole financial risk. Unlike as in [2] article, in this paper we made assumption that firms' asset value, bond value and creditworthiness criteria value are described using different correlated Wiener processes.

The structure of this paper is as follows: in Section 2 we present the model and calculations, in Section 3 we give conclusions.

2. The model

Assume that each firm has single bond (otherwise, application of following propositions requires data aggregation) and there are n assets which are traded continuously in a perfect and complete capital market. Throughout this paper, t denotes the running time variable. Also, assume that all bonds are issued at $t = 0$ and the ith bond matures at deterministic time $T_i > 0,\ i = 1, 2, \dots, n$ and $T := \min\{T_1, T_2, \dots, T_n\}$. Without any loss of the generality, we focuss on the period $[0, T]$ with a continuous trading economy. Let us consider that there exists a risk-free asset maturing not earlier than T and generating constant and non-risky annual yield r at which all market participants can borrow or lend. Assume that there are no given new bonds and any bond is not returned before the maturity, i.e., there are no jumps in the total risky bonds portfolio value and the total firms' value processes during the period $[0, T]$. Let us consider that the value of the ith firm and the value of ith bond follow geometric Brownian motion processes defined on complete probability space $(\Omega,\ \mathcal{F},\ P)$

$$\begin{cases} \mathrm{d}V_i(t) = V_i(t)(\mu_{V,i}\mathrm{d}t + \sigma_{V,i}\mathrm{d}W_{V,i}(t)), \\ \mathrm{d}D_i(t) = D_i(t)(\mu_{D,i}\mathrm{d}t + \sigma_{D,i}\mathrm{d}W_{D,i}(t)), \end{cases} \tag{1}$$

where $\mu_{D,i},\ \mu_{V,i}, \sigma_{D,i} > 0,\ \sigma_{V,i} > 0,\ i = 1, 2, \dots, n$ are constants and $\{W_{D,1}(t), \dots, W_{D,n}(t),\ \ t \geqslant 0\}$ and $\{W_{V,1}(t), \dots, W_{V,n}(t),\ \ t \geqslant 0\}$ are correlated n-dimensional Wiener processes with correlation matrices, respectively:

$$\Sigma_{D,n} = \begin{pmatrix} 1 & \rho^D_{1,2} & \dots & \rho^D_{1,n} \\ \rho^D_{1,2} & 1 & \dots & \rho^D_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^D_{1,n} & \rho^D_{2,n} & \dots & 1 \end{pmatrix} \text{ and } \Sigma_{V,n} = \begin{pmatrix} 1 & \rho^V_{1,2} & \dots & \rho^V_{1,n} \\ \rho^V_{1,2} & 1 & \dots & \rho^V_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho^V_{1,n} & \rho^V_{2,n} & \dots & 1 \end{pmatrix}.$$

and covariance $E[W_{D,i}(t)W_{V,j}(t)] = \rho^{D,V}_{i,j}t,\ i, j = 1, \dots, n$. We interpret this situation as follows: if there exists a secondary market of risky bonds then the value of risky bond can be influenced by the perturbations occurring in this secondary market[1]. The kth firm defaults at

$$\tau_k = \begin{cases} \inf\{t \geqslant 0 : V_k(t) = D_k(t)\}, \\ \infty,\ \text{if } V_k(t) \neq D_k(t). \end{cases} \tag{2}$$

After the logarithmic transformation we obtain implied Wiener processes with drift $\{Y_i(t),\ t \geqslant 0\}$

$$\mathrm{d}Y_i(t) = \mu_i\mathrm{d}t + \sigma_i\mathrm{d}W_{Y,i}(t), \tag{3}$$

where

$$Y_i(t) = \log\left(\frac{V_i(t)}{D_i(t)}\right), \quad \mu_i = \mu_{V,i} - \mu_{D,i} + \sigma^2_{D,i} - \rho^{D,V}_{i,i}\sigma_{D,i}\sigma_{V,i},$$
$$W_{Y,i} = \frac{\sigma_{V,i}W_{V,i}(t) - \sigma_{D,i}W_{D,i}(t)}{\sigma_i}, \quad \sigma_i = \sqrt{\sigma^2_{V,i} + \sigma^2_{D,i} - 2\rho^{D,V}_{i,i}\sigma_{D,i}\sigma_{V,i}}$$

[1]In practice, it is the only difference between the bond and the credit, especially in undeveloped or emerging financial markets.

with respective correlation and covariance matrices:

$$\Sigma_{Y,n} = \begin{pmatrix} 1 & \rho_{1,2} & \dots & \rho_{1,n} \\ \rho_{1,2} & 1 & \dots & \rho_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,n} & \rho_{2,n} & \dots & 1 \end{pmatrix}, \quad \hat{\Sigma}_{Y,n} = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} & \dots & \sigma_{1,n} \\ \sigma_{1,2} & \sigma_2^2 & \dots & \sigma_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1,n} & \sigma_{2,n} & \dots & \sigma_n^2 \end{pmatrix},$$

where

$$\rho_{i,j} = \frac{\sigma_{V,i}\sigma_{V,j}\rho_{i,j}^{V} - \sigma_{D,i}\sigma_{V,j}\rho_{i,j}^{D,V} - \sigma_{V,i}\sigma_{D,j}\rho_{j,i}^{D,V} + \sigma_{D,i}\sigma_{D,j}\rho_{i,j}^{D}}{\sigma_i\sigma_j},$$

$$\sigma_{i,j} = \sigma_i\sigma_j\rho_{i,j}, \ i, j = 1, 2, \dots, n.$$

The default time of kth firm is defined as stopping time of respective stochastic processes:

$$\tau_k := \inf\{t \geqslant 0 : V_k(t) = D_k(t)\} = \inf\{t \geqslant 0 : Y_k(t) = b_k\},$$

$$b_k = \log\left(\frac{D_k(0)}{V_k(0)}\right), \ k = 1, 2, \dots, n.$$

Assume that the criteria of the kth firm credibility follows a geometric Wiener process

$$\mathrm{d}A_k(t) = A_k(t)(\mu_{A,k}\mathrm{d}t + \sigma_{A,k}\mathrm{d}W_{A,k}(t)).$$

$\{W_A(t) = (W_{A,1}(t), W_{A,2}(t), \dots, W_{A,n}(t)), \ t \geqslant 0\}$ is uncorrelated with other processes $\{W_D(t), \ t \geqslant 0\}$ and $\{W_V(t), \ t \geqslant 0\}$. Also, assume that the initial condition $A_k(0) \geqslant D_k(0)$ holds. Then the value of correlated firm[2] asset is given in the following theorems:

THEOREM 2.1. *For $T \in [0, \infty)$ and geometric Wiener processes $\{D_k(t), \ t \geqslant 0\}$ and $\{A_k(t), \ t \geqslant 0\}$, $k = 1, \dots, n$ such that $D_k(0) \leqslant \min(A_k(0), V_k(0))$ the present value of firm asset at maturity is*

$$\begin{aligned} &E\left(V_1(T), V_1(T) \geqslant A_1(T), V_2(T) \geqslant A_2(T), \dots, V_n(T) \geqslant A_n(T), \tau_k \leqslant T\right) \\ &\quad = \mathrm{e}^{-rT} V_1(0) \exp\left\{2b_k\sigma_k^{-2}(\mu_k + \sigma_{1,k}) + \left(\mu_1 + \frac{1}{2}\sigma_1^2\right)T\right\}, \\ &\Phi\Bigg(\frac{\log\frac{V_1(0)}{A_1(0)} + (\mu_1 + \sigma_1^2)T}{\sigma_1\sqrt{T}} + \frac{2b_k\rho_{1,k}}{\sigma_k\sqrt{T}}, \frac{\log\frac{V_2(0)}{A_2(0)} + (\mu_2 + \sigma_{1,2})T}{\sigma_2\sqrt{T}} + \frac{2b_k\rho_{2,k}}{\sigma_k\sqrt{T}}, \\ &\quad \dots, \frac{\log\frac{V_n(0)}{A_n(0)} + (\mu_n + \sigma_{1,n})T}{\sigma_n\sqrt{T}} + \frac{2b_k\rho_{n,k}}{\sigma_k\sqrt{T}}; \Sigma_{Y,n}\Bigg), \end{aligned} \tag{4}$$

[2]Without any loss of the generality assume that it is a first firm.

where $\Phi(x_1, \dots, x_n; \Sigma)$ *denotes a cumulative distribution function of multidimensional standard normal random variable with correlation matrix* Σ.

The impact of correlation of processes $\{D_k(t),\ t \geqslant 0\}$ and $\{V_k(t),\ t \geqslant 0\}$, $k = 1, \dots, n$ to the asset value, by Theorem 2.1, in case when $n = 2$, $V_1(0) = 100$, $r = 5\%$, $\mu_k = 0.025$, $\sigma_k = 2$, $b_k = \log\frac{1}{3}$, $\log\frac{V_k(0)}{A_k(0)} = \log\frac{3}{2}$, $\rho_{1,k} = \rho_{2,k}$ and $T = 1$ is presented in Table 1.

This theorem allows to calculate the firm value in case when any other kth, $k = 2, \dots, n$, firm defaults until maturity T. In more general case, when not one but all remaining $n-1$ firms have experienced default before maturity T the following theorem holds:

THEOREM 2.2. *For* $T \in [0, \infty)$, *and geometric Wiener processes* $\{D_k(t),\ t \geqslant 0\}$ *and* $\{A_k(t),\ t \geqslant 0\}$, $k = 1, \dots, n$ *such that* $D_k(0) \leqslant \min(A_k(0), V_k(0))$ *and* $\rho_{i,j} = 0$, $i \neq j \in \{2, 3, \dots, n\}$ *the present firm value at maturity is*

$$\begin{aligned}
&E\big(V_1(T), V_1(T) \geqslant A_1(T), V_2(T) \geqslant A_2(T), \dots, V_n(T) \geqslant A_n(T),\\
&\quad \tau_2 \leqslant T, \tau_3 \leqslant T, \dots, \tau_n \leqslant T\big)\\
&\quad = \mathrm{e}^{-rT} V_1(0) \exp\left\{2\sum_{i=2}^{n} b_i \sigma_i^{-2}\left(\mu_i + \sigma_{1,i}\right) + \left(\mu_1 + \frac{1}{2}\sigma_1^2\right) T\right\}\\
&\Phi\left(\frac{\log\frac{V_1(0)}{A_1(0)} + (\mu_1 + \sigma_1^2)T}{\sigma_1\sqrt{T}} + 2\sum_{i=2}^{n} \frac{b_i \rho_{i,1}}{\sigma_i\sqrt{T}}, \frac{\log\frac{V_2(0)}{A_2(0)} + (\mu_2 + \sigma_{1,2})T}{\sigma_2\sqrt{T}} + \frac{2b_2}{\sigma_2\sqrt{T}},\right.\\
&\quad \left.\dots, \frac{\log\frac{V_n(0)}{A_n(0)} + (\mu_n + \sigma_{1,n})T}{\sigma_n\sqrt{T}} + \frac{2b_n}{\sigma_n\sqrt{T}}; \tilde{\Sigma}_{Y,n}\right).
\end{aligned} \tag{5}$$

Table 1. The value of correlated firm asset in case of Theorem 2.1

Implied correlation	**Firm's asset value**
-0.75	617.396641
-0.5	227.169655
-0.25	79.054997
0	26.053990
0.25	8.060543
0.5	2.320376
0.75	0.621617

The special case of correlation matrix $\Sigma_{Y,n}$ where $\rho_{i,j} = 0,\ i \neq j \in \{2, 3, \dots, n\}$

$$\tilde{\Sigma}_{Y,n} = \begin{pmatrix} 1 & \rho_{1,2} & \dots & \rho_{1,n} \\ \rho_{1,2} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,n} & 0 & \dots & 1 \end{pmatrix}$$

is used in last expression.

This formula can be applicable for calculating the firm asset value in case when all remaining $n-1$ firms have experienced default before maturity T. Next, we give special case of the previous theorem when $m,\ 1 < m < n$ firms default.

The impact of correlation of processes $\{D_k(t),\ t \geqslant 0\}$ and $\{V_k(t),\ t \geqslant 0\}$, $k = 1, \dots, n$ to the asset value, by Theorem 2.2, in case when $n = 3,\ V_1(0) = 100,\ r = 5\%,\ \mu_k = 0.025,\ \sigma_k = 2,\ b_k = \log\frac{1}{3},\ \log\frac{V_k(0)}{A_k(0)} = \log\frac{3}{2},\ \rho_{1,2} = \rho_{1,3}$ and $T = 1$ is presented in Table 2.

COROLLARY 2.1. *For $T \in [0, \infty)$, Brownian motions $\{D_k(t),\ t \geqslant 0\}$ and $\{A_k(t),\ t \geqslant 0\}$ such that $D_k(0) \leqslant \min(A_k(0), V_k(0))$, $k = 1, 2, \dots, m$, $m < n$ and $V_l(0) = D_l(0)$, $l = m+1, m+2, \dots, n$ and $\rho_{i,j} = 0$, $i \neq j \in \{2, 3, \dots, m\}$ the present value of firm asset at maturity is*

$$\begin{aligned}
&E\big(V_1(T), V_1(T) \geqslant A_1(T), V_2(T) \geqslant A_2(T), \dots, V_n(T) \geqslant A_m(T),\\
&\quad \tau_2 \leqslant T, \tau_3 \leqslant T, \dots, \tau_m \leqslant T\big)\\
&\quad = \mathrm{e}^{-rT} V_1(0) \exp\left\{2\sum_{i=2}^{m} b_i \sigma_i^{-2}(\mu_i + \sigma_{1,i}) + \left(\mu_1 + \frac{1}{2}\sigma_1^2\right)T\right\},\\
&\Phi\Bigg(\frac{\log\frac{V_1(0)}{A_1(0)} + (\mu_1 + \sigma_1^2)T}{\sigma_1\sqrt{T}} + \sum_{i=2}^{m} \frac{2b_i\rho_{i,1}}{\sigma_i\sqrt{T}}, \frac{\log\frac{V_2(0)}{A_2(0)} + (\mu_2 + \sigma_{1,2})T}{\sigma_2\sqrt{T}} + \frac{2b_2}{\sigma_2\sqrt{T}},\\
&\quad \dots, \frac{\log\frac{V_m(0)}{A_m(0)} + (\mu_m + \sigma_{1,m})T}{\sigma_m\sqrt{T}} + \frac{2b_m}{\sigma_m\sqrt{T}}; \tilde{\Sigma}_{Y,m}\Bigg).
\end{aligned} \tag{6}$$

The proofs of these propositions are based on the reflection principle of multidimensional correlated Wiener process.

The stochastic boundaries of credibility criteria in both theorems $\{A_k(t),\ t \geqslant 0\}$, $k = 1, 2, \dots, n$ can be treated as some reserve (upper credibility level) of financial institution in assessment of credit risk of bonds portfolio. One of the conditions of these theorems is $D_k(0) \leqslant \min(A_k(0), V_k(0))$, and when the trajectories of process $\{V_k(t),\ t \geqslant 0\}$ hit the default boundary $\{D_k(t),\ t \geqslant 0\}$ it means the default of the k th firm. In this case, the financial intitution follows so called *principle of conservatism* in assessing the credit risk. To implement these formulas in practice, it is necessary to

Table 2. The value of correlated firm asset in case of Theorem 2.2

Implied correlation	**Firm's asset value**
-0.75	161.479539
-0.5	186.874686
-0.25	173.586646
0	130.407991
0.25	79.993750
0.5	40.315704
0.75	16.282379

aggregate other data related to other bonds in such way that the value of all assets of firm could be interpreted as the value of one asset.

3. Conclusion

We give the tools of evaluation of firm's asset value in case when the processes, describing the firms' asset values and bonds are correlated. In adition, we defined an implied correlation and showed numerically itd impact to the firm's asset value in both cases. Also, we think that presented model is useful for assessment the credit risk with respect to other sources of financial risks, notably, operational risk and the risk of market change.

References

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REZIUMĖ

M. Valužis. Dabartinė įmonės turto vertė susijusių paskolų atveju: struktūrinių kredito rizikos modelių apibendrinimas

Šiame straipsnyje apibendrinami struktūriniai kredito rizikos modeliai tuo atveju, kai paskolų portfelį sudaro $n \geqslant 2$ susijusių paskolų. Įmonių turto ir paskolų vertės yra apibrėžiamos susijusiais geometriniais Brown'o judesiais. Nustatant dabartinę įmonės turto vertę taikoma apibendrintų pasirinkimo sandorių (angl. *contingent option*) verčių apskaičiavimo technika kredito rizikos kontekste. Tokiu būdu šiame modelyje taip pat atsižvelgiama į rinkos pokyčių ir operacinę riziką. Paskolas paėmusių įmonių nemokumo koreliacija apibrėžiama naudojantis visomis įmanomomis Brown'o judesių koreliacijomis. Taip pat skaitiškai parodoma apibendrintos koreliacijos įtaka paskolą paėmusios įmonės dabartinei turto vertei.