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Central limit theorem for alternating renewal processes

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Abstract. Functional central limit theorems for stationary alternating renewal processes with dependent work and repair times, and for associated workload processes are stated. The weak convergence of distributions of properly scaled processes in the Skorokhod space holds under some regularity condition imposed on the distribution functions of work and repair times.

Keywords: Skorokhod space, central limit theorem, alternating renewal processes.

An ordinary renewal process is defined by a sequence $\{X_i,\ i = 1, 2, ...\}$ of independent identically distributed non-negative random variables which can be interpreted as lifetimes (or work periods) of an element (machine, system). The sequence $T_n = X_1 + ... + X_n$ is called a renewal process, and T_n are called renewal times. Renewal process can also be described by giving a counting process

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}\{\mathbf{T}_n \leqslant t\},$$

where $\mathbf{1}\{...\}$ denotes the indicator function. $N(t)$ is called a counting renewal process or just a renewal process. The sequence $\{T_n\}$ and the counting process $N(t)$ are related by

$$\{N(t) \geqslant n\} = \{T_n \leqslant t\}.$$

In this scheme, the end of the lifetime of an element coincides with the begining of the lifetime of the next element. In other words, the elements are replaced instantly (the repair times equal zero). However, in real situations often there is an installation (or repairing) period between the working periods. A more realistic model is alternating renewal process defined by a sequence of independent identically distributed random vectors $\{(X_i, Y_i),\ i = 1, 2, ...\}$ where components $Y_i,\ i = 1, 2, ...$ represent repair times (or waiting times). The alternating renewal process might be a suitable model for a system that can be in one of two stages, ON or OFF (a machine alternately working and under repair, or a workstation transmitting data and remaining silent, or a queueing system performing service and being idle).

A stationary alternating renewal process with dependent ON and OFF periods was constructed in [1] in the following way. Let $\{(X, Y), (X_i, Y_i), i = 1, 2, ...\}$ be an i.i.d. sequence of vectors with positive components which may be arbitrarily dependent. We assume $\mu = EX < \infty$ and $\nu = EY < \infty$. Denote

$$Z = X + Y, \quad Z_i = X_i + Y_i, \ i \geqslant 1, \quad T_n = T_0 + \sum_{i=1}^{n} Z_i,$$

where $T_0 = X_0 + Y_0$ is a delay random variable introduced to make the renewal sequence stationary. The random vector (X_0, Y_0) is independent of the sequence (X, Y), and its distribution is defined by

$$P\{X_0 = 0,\ Y_0 \in \mathrm{d}y\} = \lambda^{-1} P\{Y > y\}\,\mathrm{d}y, \quad (y > 0),$$
$$P\{X_0 \in \mathrm{d}x,\ Y_0 \in \mathrm{d}y\} = \lambda^{-1} P\{X > x,\ Y \in \mathrm{d}y\}\,\mathrm{d}x, \quad (x > 0,\ y > 0),$$

where

$$\lambda = EZ = \mu + \nu.$$

Each T_n is interpreted as the begining of the working period (or the end of the waiting period), and $T_n + X_{n+1}$ is the end of the working period (or the begining of the next waiting or repairing period). With the alternating renewal process we associate two stationary processes, the ON/OFF process $A(t)$ defined by

$$A(t) = \mathbf{1}\{X_0 > t\} + \sum_{n=0}^{\infty} \mathbf{1}\{T_n \leqslant t < T_n + X_{n+1}\} \quad (t \geqslant 0)$$

and the workload process

$$B(t) = \int_0^t A(s)\,\mathrm{d}s.$$

The process $A(t)$ is a binary process with $A(t) = 1$ if t falls into a working period and $A(t) = 0$ if t is in a repairing period. Stationarity implies

$$EA(t) = P\{A(t) = 1\} = \mu\lambda^{-1}.$$

Let

$$Z_n(t) = \frac{\sum_{i=1}^{n} (A_i(t) - \mu\lambda^{-1})}{\sqrt{n}},$$

where $A_i(t)$ are i.i.d. copies of the process $A(t)$.

An arbitrary distribution function $F(t)$ with $F(t) = 0$ for $t \leqslant 0$ will be said to satisfy condition (F) if

$$\limsup_{t \to 0} \frac{F(t) - F(0)}{t} < \infty \tag{F}$$

Note that condition (F) is weaker than the requirement for the distribution F to have a bounded density in the neighbourhood of 0.

Denote $F_X(t) = P\{X \leqslant t\}$, $F_Y(t) = P\{Y \leqslant t\}$.

THEOREM 1. *If* F_X *(or* F_Y *or both) satisfy condition* (F)*, then the sequence* Z_n *converges in the space* $\mathbf{D} = D[0,\infty)$ *endowed with the Skorokhod* J_1 *– topology toward a zero-mean stationary Gaussian process with almost surely continuous paths and the covariance function of* $A(t)$*.*

Proof. By virtue of the CLT, the finite-dimensional distributions of the process Z_n converge to the multidimensional Gaussian distributions. Thus, it suffices to prove tightness of Z_n.

One can write

$$A_i(t) = A_i(0) + N_i^+(t) - N_i^-(t),$$

where $N_i^+(t)$ (resp. $N_i^-(t)$) is the number of upward (resp. downward) jumps of process $A_i(t)$ over time interval $(0,t]$. Therefore,

$$Z_n(t) = Z_n(0) + L_n^+(t) - L_n^-(t),$$

where

$$L_n^+(t) = \frac{\sum_{i=1}^n (N_i^+(t) - t\lambda^{-1})}{\sqrt{n}}$$

and

$$L_n^-(t) = \frac{\sum_{i=1}^n (N_i^-(t) - t\lambda^{-1})}{\sqrt{n}}.$$

Each point process $N_i^+(t)$ is a stationary renewal process with interrenewal time distribution function satisfying condition (F). By Theorem 7.2.3 in [2], (cf. Theorem 2 in [3],Theorem 5.3.1 in [4]) sequence L_n^+ is tight and converges in the space $\mathbf{D}$ toward a Gaussian process with continuous paths. Interrenewal intervals of the process $N_i^-(t)$ are $(T_j - Y_j, T_j + X_{j+1})$, so interrenewal times form a 1-dependent sequence of identically distributed random variables with distrubution function satisfying condition (F). Using similar arguments as in [2]–[4] one can show that sequence L_n^- is tight and converges to a Gaussian process. Since the limits of both sequences have a.s. continuous paths, the limit of Z_n has a.s. continuous paths. Stationarity of processes Z_n ensures stationarity of the limiting Gaussian process ⊡

Put

$$B(t) = \int_0^t A(s)\,\mathrm{d}s.$$

Process $B(t)$ can be interpreted as the total working time in the time interval $[0,t]$. Let

$$W_n(t) = \frac{\sum_{i=1}^n (B_i(t) - \mu\lambda^{-1}t)}{\sqrt{n}},$$

where $B_i(t)$ are i.i.d. copies of the process $B(t)$.

THEOREM 2. *Under the conditions of Theorem* 1*, the sequence* W_n *converges in the space* $\mathbf{D} = D[0,\infty)$ *endowed with the Skorokhod* J_1 *– topology toward a zero-mean continuous Gaussian process with stationary increments.*

This theorem can be proved using the same arguments as in proof of Proposition 5.3.6 in [4].

References

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REZIUMĖ

R. Banys. Alternuojančiųjų atstatymo procesų centrinė ribinė teorema

Suformuluota alternuojančių atstatymo procesų funkcinė centrinė ribinė teorema, kai darbo ir atstatymo periodai yra priklausomi, o šių periodų pasiskirstymo funkcijos atitinka tam tikrą reguliarumo sąlygą.