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Proof-search of propositional intuitionistic logic sequents by means of classical logic calculus

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Abstract. In the paper, we define some classes of sequents of the propositional intuitionistic logic. These are classes of primarily and α-primarily reducible sequents. Then we show how derivability of these sequents in a propositional intuitionistic logic sequent calculus LJ_0 can be checked by means of a propositional classical logic sequent calculus LK_0.

Keywords: Glivenko theorem, classical propositional sequent calculus, intuitionistic propositional sequent calculus.

1. Introduction

In the paper, we define some classes of sequents of the propositional intuitionistic logic. These are classes of primarily and α-primarily reducible sequents. Then we show how derivability of these sequents in a propositional intuitionistic logic sequent calculus LJ_0 can be checked by means of a propositional classical logic sequent calculus LK_0. The paper is organized as follows. First, we introduce the calculi LK_0 and LJ_0. Then, we define the class of primarily reducible sequents. Further we modify LK_0 and LJ_0 and introduce the class of α-primarily reducible sequents. In the end, we define a subclass of α-reducible sequents by introducing some restriction on syntax of sequents.

2. Calculi LK_0 and LJ_0

Calculus LK_0 is a variant of the classical propositional Gentzen-like sequent calculus. It is defined as follows:

1. Axioms: $\Gamma, E \to E, \Delta$.

2. Rules:

$$\frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\ (\wedge \to), \qquad \frac{\Gamma \to A, \Delta;\ \Gamma \to B, \Delta}{\Gamma \to A \wedge B, \Delta}\ (\to \wedge),$$

$$\frac{A, \Gamma \to \Delta;\ B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ (\vee \to), \qquad \frac{\Gamma \to A, B, \Delta}{\Gamma \to A \vee B, \Delta}\ (\to \vee),$$

$$\frac{\Gamma \to A, \Delta}{\Gamma, \neg A \to \Delta}\ (\neg\to), \qquad \frac{\Gamma, A \to \Delta}{\Gamma \to \neg A, \Delta}\ (\to\neg),$$

$$\frac{\Gamma \to A, \Delta;\ B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ (\supset\to), \quad \frac{\Gamma, A \to B, \Delta}{\Gamma \to A \supset B, \Delta}\ (\to\supset).$$

Here: E denotes an atomic formula; A and B denote arbitrary formulas; Γ and Δ denote finite, possibly empty, multisets of formulas.

LJ_0 is a variant of the intuitionistic propositional Gentzen-like sequent calculus. It is obtained from LK_0 by the following changes. There is at most one formula in the succedent. Thus, $\Delta = \emptyset$ in the succedent rules. Also.

Rule $(\to\vee)$ is replaced by the following one:

$$\frac{\Gamma \to A \ \text{ or } \ B}{\Gamma \to A \vee B}(\to\vee).$$

Rule $(\neg\to)$ is replaced by

$$\frac{\Gamma, \neg A \to A}{\Gamma, \neg A \to \Delta}(\neg\to).$$

Rule $(\supset\to)$ is replaced by

$$\frac{\Gamma, A \supset B \to A;\ B, \Gamma \to \Delta}{\Gamma, A \supset B \to \Delta}(\supset\to).$$

We introduce here some notation. We denote a derivation tree by V and the height of the derivation tree by $h(V)$. The height of a derivation tree is reckoned to be the length of the longest branch in it. The length of a branch is measured by the number of rule applications in it.

Now we present some well known properties of LK_0 and LJ_0. All LK_0 rules are strongly invertible. I.e., if the conclusion is derivable, then also is the/each premise; moreover, there exists a derivation of the/each premise such that its height is less or equal than that one of the conclusion.

All LJ_0 rules, except $(\to\vee)$, $(\neg\to)$, and $(\supset\to)$, are strongly invertible. $(\supset\to)$ is strongly invertible with respect to the right premise. I.e., $\vdash^V \Gamma, A \supset B \to \ \Delta$ implies the existence of V' such that $\vdash^{V'} \Gamma, B \to \Delta$ and $h(V') \leqslant h(V)$.

The following properties hold for both LK_0 and LJ_0. Any sequent of the shape $\Gamma, D \to D, \Delta$ is derivable (D any formula). The rules of weakening and contraction are strongly admissible. The rule of cut is admissible. The calculi are correct and complete.

We will freely apply these properties further.

3. Primary sequents

Glivenko proved in [1] that a formula beginning with '$\neg$' is derivable in a classical propositional logic calculus iff it is derivable in its intuitionistic counterpart. Due to rule invertibility, a sequent $A_1, \ldots, A_n \to$ is derivable in LK_0 iff the sequent $\to \neg(A_1 \wedge \ldots \wedge A_n)$ is derivable in LK_0. According to the Glivenko theorem, the last sequent is derivable in LK_0 iff it is derivable in LJ_0. Thus, we have that a sequent with the empty succedent is derivable in a classical propositional logic calculus iff it is derivable in its intuitionistic counterpart. See also [2].

A sequent of the shape $\Pi, \neg\Gamma \rightarrow \Theta$ is called primary. Here Π is the empty set or a multiset consisting of atomic formulas; $\neg\Gamma$ is the empty set or a multiset of formulas each of which is preceded by '$\neg$'; Θ is the empty set or an atomic formula.

If $\Theta = \emptyset$ or $\Pi \cap \Theta \neq \emptyset$, then a primary sequent $S = \Pi, \neg\Gamma \rightarrow \Theta$ is derivable in LK_0 iff it is derivable in LJ_0. If $\Theta = E$, then only $(\neg \rightarrow)$ is applicable to S. But then Θ is dropped, and $LJ_0 \vdash \Pi, \neg\Gamma \rightarrow \Theta$ iff $LJ_0 \vdash \Pi, \neg\Gamma \rightarrow$. The last sequent, by the Glivenko theorem, is derivable in LJ_0 iff it is derivable in LK_0. Thus, in this case, we have that $LJ_0 \vdash \Pi, \neg\Gamma \rightarrow \Theta$ iff $LK_0 \vdash \Pi, \neg\Gamma \rightarrow$. E.g., instead of considering $\neg\neg A \rightarrow A$ in LJ_0, we can consider $\neg\neg A \rightarrow$ in LK_0 (A atomic).

We denote a derivation tree with a sequent S at the bottom by $V(S)$.

A sequent S is called primarily reducible iff there exists an LJ_0 derivation tree $V(S)$ such that only invertible rules of LJ_0 are applied in it (i.e., any rule except $(\neg \rightarrow)$, $(\supset\rightarrow)$, and $(\rightarrow \vee)$) and each leaf of which is an axiom, a primary sequent, or a sequent with the empty succedent. Such a tree is called a primary reduction tree. E.g., any sequent of the shape $\Gamma \rightarrow \neg A$ is primarily reducible: applying $(\rightarrow \neg)$ to this sequent, we get the primary reduction tree.

Suppose that each leaf of a primary reduction tree $V(S)$ is of the shape $\Gamma, D \rightarrow D, \Delta$ (D any formula) or a sequent with the empty succedent. Then each such a leaf is derivable in LJ_0 iff it is derivable in LK_0. Note also that the invertible rules of LJ_0 coincide with the corresponding ones of LK_0 for sequents with at most one formula in the succedent. Due to the fact that all rules of LK_0 are invertible, rule application order has no impact on derivability in LK_0. Therefore, $LJ_0 \vdash S$ iff $LK_0 \vdash S$.

If a primary reduction tree $V(S)$ has a non-axiom leaf with an atom E in the succedent, then E must be removed before we can consider the leaf in LK_0. Therefore, in this case, though we use only invertible rules, we cannot consider S directly in LK_0 in order to check if it is derivable in LJ_0. We construct the reduction tree first, then replace each non-axiom leaf of the shape $\Pi, \neg\Gamma \rightarrow E$ by $\Pi, \neg\Gamma \rightarrow$, and then consider the non-axiom leaves in LK_0.

4. Modifications of LK_0 and LJ_0

In this section, we use the ideas of [3] and [4]. We mention also [5] and [6]. Let LK_0' and LJ_0' be the calculi obtained from LK_0 and LJ_0, respectively, by making the restriction that A in the explicit $A \supset B$ in the rule $(\supset\rightarrow)$ is not atomic and by introducing a new derivation rule:

$$\frac{E, B, \Gamma \rightarrow \Delta}{E, E \supset B, \Gamma \rightarrow \Delta}(E \supset\rightarrow).$$

Here E is atomic. This rule corresponds to the $(\supset\rightarrow)$ rule with the exception that the left premise $E, \Gamma \rightarrow E, \Delta$ (calculus LK_0) is dropped. Note that $(E \supset\rightarrow)$ is strongly invertible because $(\supset\rightarrow)$ is strongly invertible with respect to the right premise in both LK_0 and LJ_0.

$LK_0'' = LK_0' \cup LK_0$ and $LJ_0'' = LJ_0' \cup LJ_0$.

By, e.g., $(\rightarrow A \wedge B)$, we denote an application of $(\rightarrow \wedge)$ with $A \wedge B$ as the main formula, etc.

LEMMA 4.1. *Let $Calc \in \{LK_0'', LJ_0''\}$ and $Calc \vdash^V S$, where S is any sequent. Suppose further that the first rule applied in V counting from the bottom is $(E \supset D \rightarrow)$, where E is atomic, and there are no other applications of this shape in V. Then there exists V' such that $Calc \vdash^{V'} S$ and V' is free of rule applications of the type $(E \supset D \rightarrow)$ (E any atomic, D arbitrary).*

LEMMA 4.2. *Let $Calc \in \{LK_0, LJ_0\}$ and S be an arbitrary sequent. $Calc \vdash S$ iff $Calc' \vdash S$.*

Proof. 1) $Calc' \vdash S \Rightarrow Calc \vdash S$. This is obvious.

2) $Calc \vdash^V S \Rightarrow Calc' \vdash S$.

First, we prove that $Calc'' \vdash^V S \Rightarrow Calc' \vdash S$. For the proof, we use induction on the number of $(E \supset D \rightarrow)$ type applications in V. The base case is obvious. The inductive case is considered as follows. We take an $(E \supset D \rightarrow)$ application in V above which there are no other such applications and apply the previous lemma, reducing the number of $(E \supset D \rightarrow)$ applications. It remains to apply the inductive hypothesis.

We have: $Calc \vdash^V S \Rightarrow Calc'' \vdash S \Rightarrow Calc' \vdash S$.

5. α-primary sequents

Using the results of the previous section, we will expand the class of primary sequents. Due to Lemma 4.2, as far as the sequent derivability is concerned, LK_0' and LJ_0' can be freely interchanged with LK_0 and LJ_0, respectively.

A sequent of the shape $\Pi, (E \supset D)_i, \neg\Gamma \rightarrow \Theta$, $i \geqslant 0$, is called α-primary. Here $(E \supset D)_i$ is the empty set or a multiset: $E_1 \supset D_1, E_2 \supset D_2, \ldots, E_m \supset D_m$, where E_i are atomic and D_i arbitrary formulas; Π is the empty set or a multiset consisting of atomic formulas and $E_i \notin \Pi$; Θ is the empty set or an atomic formula.

If $\Theta = \emptyset$ or $\Pi \cap \Theta \neq \emptyset$, then an α-primary sequent $S = \Pi, (E \supset D)_i, \neg\Gamma \rightarrow \Theta$ is derivable in LJ_0 iff it is derivable in LK_0.

If $\Theta = E$, then only $(\neg \rightarrow)$ is applicable to S in LJ_0'. But then Θ is dropped, and $LJ_0' \vdash \Pi, (E \supset D)_i, \neg\Gamma \rightarrow \Theta$ iff $LJ_0' \vdash \Pi, (E \supset D)_i, \neg\Gamma \rightarrow$. The last sequent, by the Glivenko theorem, is derivable in LJ_0 iff it is derivable in LK_0. We have that $LJ_0 \vdash \Pi, (E \supset D)_i, \neg\Gamma \rightarrow \Theta$ iff $LK_0 \vdash \Pi, (E \supset D)_i, \neg\Gamma \rightarrow$.

A sequent S is called α-primarily reducible iff there exists an LJ_0' derivation tree $V(S)$ such that only invertible rules of LJ_0' are applied in it and each leaf of which is an axiom, an α-primary sequent, or a sequent with the empty succedent. Such a tree is called an α-primary reduction tree.

Suppose that each leaf of an α-primary reduction tree $V(S)$ is of the shape $\Gamma, D \rightarrow D, \Delta$ (D any formula) or a sequent with the empty succedent. Then each such a leaf is derivable in LJ_0' iff it is derivable in LK_0'. Note also that the invertible rules of LJ_0' coincide with the corresponding ones of LK_0' for sequents with at most one formula in the succedent. Due to the fact that all rules of LK_0' are invertible, rule application order has no impact on derivability in LK_0. Therefore, $LJ_0 \vdash S$ ($LJ_0' \vdash S$) iff $LK_0 \vdash S$ ($LK_0' \vdash S$).

If an α-primary reduction tree $V(S)$ has a non-axiom leaf with some atomic formula E in the succedent, then E must be dropped before we can consider the leaf in LK_0'. Therefore, in this case, though we use only invertible rules, we cannot consider S directly in LK_0 in order to check if it is derivable in LJ_0. We first construct the reduction tree, then make succedents of the non-axiom leaves empty, and then consider the non-axiom leaves in LK_0 (or LK_0').

5.1. Some expansion of the class of α-primarily reducible sequents

In this section, we present some means which allows us to expand the class of α-reducible sequents. We make use of ideas of [3] and [4].

$F_1 = (B \wedge C) \supset D$ and $F_1' = \alpha \supset (\delta \supset D)$. Here $\alpha \in \{B, C\}$ and $\delta \in \{B, C\} \setminus \{\alpha\}$. $F_2 = (B \vee C) \supset D$ and $F_2' = (B \supset D) \wedge (C \supset D)$.

Convince yourself that the sequents $F_1 \to F_1'$, $F_1' \to F_1$ and $F_2 \to F_2'$, and $F_2' \to F_2$ are derivable in both LK_0 and LJ_0.

The rules

$$\frac{\alpha \supset (\delta \supset D), \Gamma \to \Delta}{(B \wedge C) \supset D, \Gamma \to \Delta}(\wedge \supset\to) \quad \text{and} \quad \frac{(B \supset D), (C \supset D), \Gamma \to \Delta}{(B \vee C) \supset D, \Gamma \to \Delta}(\vee \supset\to)$$

are admissible and invertible in LJ_0 and LK_0. To see this, use cut and the fact that the above four sequents are derivable in LJ_0 and LK_0. It follows from this and Lemma 4.2 that these rules are admissible also in LJ_0' and LK_0'.

With the help of these rules, we get that, e.g., the sequent

$$(D \wedge E) \supset B \to A$$

(E and A atomic) is α-reducible:

$$\frac{E \supset (D \supset B) \to A}{(D \wedge E) \supset B \to A}(\wedge \supset\to)$$

and the premise is α-primary.

Thus, let us redefine the notion of α-primarily reducible sequents. A sequent S is called α-primarily reducible iff there exists an $LJ_0' \cup \{(\wedge \supset\to), (\vee \supset\to)\}$ derivation tree $V(S)$ such that only invertible rules of $LJ_0' \cup \{(\wedge \supset\to), (\vee \supset\to)\}$ are applied in it and each leaf of which is an axiom, an α-primary sequent, or a sequent with the empty succedent. Such a tree is called an α-primary reduction tree.

6. Definition of a subclass of α-primarily reducible sequents

Now, let us define a subclass of α-reducible sequents by introducing some restriction on syntax of sequents.

First, we give some preparatory definitions. Indicators "formula F is negative" are: 1) F occurs in the antecedent and 2) F occurs in the scope of $\neg$ or in the left scope of $\supset$. An occurrence of a formula in a sequent S is called positive in S iff 1) it is in its succedent and there are no indicators showing that it is negative or 2) the number of indicators indicating that F is negative is even. Otherwise the occurrence is called negative in S. Let us consider an example:

$$S = \neg\neg\neg C \to \neg B.$$

C is positive and B is negative in S. $\neg B$ is positive and $\neg C$ is negative. And so on.

Suppose that G is a subformula of F and F is subformula of itself only in a sequent S. The number of alternations positive-negative (or negative-positive) obtained by "going into the depth of F" until the occurrence of G is reached is called the degree of positiveness or negativeness of the occurrence of G in S. E.g., let us take the above example. B is a subformula of $\neg B$ and the latter formula is a subformula of itself only in S. $\neg B$ is positive and B is negative in S. We have one alternation and conclude that the occurrence of B is negative of the first degree in S. In the same way, $\neg B$ is positive of the zeroth degree and C is positive of the third degree in S.

Now we are ready to define a class $\mathfrak{C}$ of α-reducible sequents. A sequent S belongs to $\mathfrak{C}$ iff

1) it has no zeroth degree negative $\supset$ in the left scope of which $\neg$ or $\supset$ occurs. I.e., there are no situations like this: $(A \supset B) \supset C \to$ or $\neg B \supset C \to$;

2) it has no zeroth degree positive $\supset$ in the left scope of which a first degree negative $\supset$ occurs in the left scope of which $\supset$ or $\neg$ occurs. I.e., there are no situations like this: $\to ((A \supset B) \supset C) \supset D$ or $\to (\neg B \supset C) \supset D$;

3) it has no positive $\vee$ of the zeroth degree.

It is easy to see that if a sequent belongs to $\mathfrak{C}$, then it is α-reducible. However, not every α-reducible sequent belongs to $\mathfrak{C}$. Such is, e.g., the sequent

$$E \supset ((D \supset B) \supset C) \to A$$

(E and A atomic).

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REZIUMĖ

R. Alonderis. Propozicinės intuicionistinės logikos sekvencijų įrodymo paieška naudojant klasikinės logikos skaičiavimą

Straipsnyje yra apibrėžtos primariškai ir alfa-primariškai redukuojamų propozicinės intuicionistinės logikos sekvencijų klasės. Parodoma kaip nustatyti šių sekvencijų įrodomumą intuicionistinės logikos skaičiavime naudojant efektyvesnį klasikinės logikos skaičiavimą.

Raktiniai žodžiai: Glivenko teorema, klasikinis propozicinis sekvencinis skaičiavimas, intuicionistinis propozicinis sekvencinis skaičiavimas.