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On calculation of recursive M- and GM-estimates in LQG control systems

Rimantas PUPEIKIS (MII)
e-mail: pupeikis@ktl.mii.lt

Abstract. The aim of the given paper is development of a parametric identification approach for a closed-loop system when the parameters of a discrete-time linear time-invariant (LTI) dynamic system as well as that of LQG (Linear Quadratic Gaussian) controller are not known and ought to be calculated. The recursive techniques based on an the maximum likelihood (M) and generalized maximum likelihood (GM) estimator algorithms are applied here in the calculation of the system as well as noise filter parameters. Afterwards, the recursive parameter estimates are used in each current iteration to determine unknown parameters of the LQG-controller, too. The results of numerical simulation by computer are discussed.

Keywords: LQG control systems, closed-loop, identification, outliers.

1. Introduction

The stochastic optimal control of a discrete-time LTI dynamic system is performed using the LQG approach [1]. In designing a robust control system, one ought to determine the type of uncertainties appearing in the system to be controlled [6]. On the other hand, there are many types of uncertainties in system description models. One of the main ones of them is the uncertainty arising in the output disturbance description of a plant model to be used. It is assumed frequently that output of the system is affected by Gaussian disturbance. However, nonnormal noise, and particularly the presence of outliers, degrades the performance of a system acting in a closed-loop. Therefore ordinary recursive techniques used for a parametric identification of LQG control systems, as a rule, are inefficient. In such a case, robust recursive techniques ought to be applied here.

In what follows, we introduce the robust recursive GM- and M- procedures for calculating robust estimates of the parameters of LTI dynamic systems, acting in a closed-loop in the case of correlated noise with outliers in it. Note that the class of GM-estimators contains a class of maximum likelihood type estimators. The class of GM-estimators is defined implicitly by the first order condition

$$\sum_{t=1}^{N} \mathbf{x}(t)\zeta\left\{\mathbf{x}(t), [y(t) - \mathbf{x}^T(t)\theta]/\sigma\right\} = 0. \tag{1}$$

Here $\mathbf{x}(t)$ is the set of regressors, σ denotes the scale of residuals $\mathbf{n}(t)$ of the linear regression model $y(t) = \mathbf{x}^T(t)\theta + \delta(t)$, $t = 1, \ldots, N$ where θ is a vector of unknown parameters. The function $\zeta\{\cdot,\cdot\}$ in (1) depends on both the set of regressors $\mathbf{x}(t)$ and the standardized residual $\delta(t)/\sigma$. The conditions that ought to be satisfied by $\zeta\{\cdot,\cdot\}$ in order that the GM-estimator have nice asymptotic properties are known in advance [4]. The ordinary least-squares estimator could be obtained as a special case of (1) by setting in it the function $\tau(\mathbf{x}(t),r)=r^2/2$ with $\partial\tau(\mathbf{x}(t),r)/\partial r=\zeta\{\mathbf{x}(t),r\}$, where r is a short form of the standardized residual. In such a case, the class of M-estimators is obtained by setting $\tau(\mathbf{x}(t),r)=\rho(r)$, with $d\rho(r)/dr=\psi(r)$. Various $\psi(\cdot)$ functions lead to various M-estimates.

2. The Statement of the Problem

Assume that a control system to be observed is causal, linear, and time-invariant with one output $\{y(k)\}$ and one input $\{u(k)\}$, expressed by the equation

$$y(k)=G_0(q^{-1};\ \theta)u(k)+\underbrace{H_0(q^{-1};\varphi)\xi(k)}_{v(k)}, \tag{2}$$

that consists of two parts (Fig. 1): a system model $G_0(q^{-1};\theta)$ and a noise model $H_0(q^{-1};\varphi)$. Here k is the current number of observations of a respective signal, θ,φ are unknown parameter vectors to be estimated, q^{-1} is the backward time-shift operator such that $q^{-1}u(k)=u(k-1)$, $\{\xi(k)\}$ is used to generate unmeasurable noise $\{v(k)\}$.

The aim of the given paper is to estimate the parameter vector θ of the LTI system $G_0(q^{-1};\ \theta)$, acting in the closed-loop simultaneously with the current parameter vector α of the LQG controller $G_R(q^{-1};\alpha)$, by observations $\{u(k),y(k)\}$ $\forall k=1,2,\ldots,$ in the case of additive correlated noise $\{v(k)\}$, that contains large outliers and corrupts the output $\{y(k)\}$ of the system.

3. Identification in the presence of outliers

Given the model (Fig. 1) and data $\mathbf{Z}^N=\{u(1),\ldots,u(N),y(1),\ldots,y(N)\}$ and assuming that the white noise $\{\xi(k)\}$, $k=1,2,\ldots$ is really a sequence of independent identically distributed variables with an ϵ-contaminated distribution of the form $p(\xi(k))=(1-\epsilon)N(0,\sigma_\mu^2)+\epsilon N(0,\sigma_\varsigma^2)$, and the variance $\sigma_\xi^2=(1-\epsilon)\sigma_\mu^2+\epsilon\sigma_\varsigma^2$, one can determine the prediction error estimate $\hat{\theta_N}$ of the parameter vector $\theta^T=(\mathbf{a}^T,\mathbf{b}^T)=(a_1,\ldots,a_m,b_0,b_1,\ldots,b_m)$ by minimizing $\hat{\theta}_N=arg\min_{\theta\in D_M}\tilde{V}_N(\theta,\mathbf{Z}^N)$ with

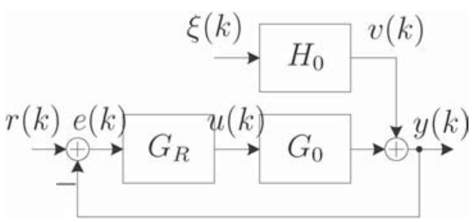

Fig. 1. A closed-loop system to be observed. Here $G_R\equiv G_R(q^{-1};\alpha)$ $G_0\equiv G_0(q^{-1};\theta)$, and $H_0\equiv H_0(q^{-1};\varphi)$.

$\tilde{V}_N(\theta, \mathbf{Z}^N) = \frac{1}{N}\sum_{k=1}^{N}\rho(e_F(k,\theta/s))$, or by solving the equation $\sum_{t=1}^{N}\mathbf{z}(t)\{\psi[y(t) - \mathbf{z}^T(t)\theta]\} = 0$, in the vector form. Here $p\{\xi(k)\}$ is the probability density distribution of the sequence $\{\xi(k)\}, k = 1, 2, \ldots$; $\xi(k) = (1-\gamma_k)\mu_k + \gamma_k\varsigma_k$ is the value of the sequence $\{\xi(k)\}, k = 1, 2, \ldots$ at a time moment k; γ is a random variable, taking values 0 or 1 with probabilities $p(\gamma_k = 0) = 1 - \epsilon$, $p(\gamma_k = 1) = \epsilon$; μ_k, ς_k are sequences of independent Gaussian variables with zero means and variances $\sigma_\mu^2, \sigma_\varsigma^2$, respectively; besides, $\sigma_\mu < \sigma_\varsigma$; $0 \leqslant \epsilon \leqslant 1$ is the unknown fraction of contamination; $\hat{\theta}_N$ is the robust estimate of the parameter vector θ, established by processing N pairs of input-output samples; s is the scale of residual; $\rho(\cdot)$ is a real-valued function that is even and non-decreasing for positive residuals, and $\rho(0) = 0$, $\psi = \rho'$.

To get a better performance of $\hat{\theta}_N$ in the case of very long-tailed distributions, a function $\tilde{V}_N(\theta, \mathbf{Z}^N)$ satisfying $\psi(x) = 0$, if $| x |> c_H$, for some $c_H > 0$ could be selected.

The current M- estimates of an unknown vector of the parameters θ of LTI system with $G(q,\theta)$ of the form

$$G_0(q^{-1};\theta) = \frac{B(q^{-1};\mathbf{b})}{A(q^{-1};\mathbf{a})} = \frac{b_0 + b_1q^{-1} + b_2q^{-2} + \cdots + b_mq^{-m}}{1 + a_1q^{-1} + \cdots + a_mq^{-m}} \tag{3}$$

according to [5] can be calculated using three techniques: the S-algorithm, the H-algorithm, and the W-one:

$$\hat{\theta}(k) = \hat{\theta}(k-1) + \frac{\Gamma(k-1)\mathbf{z}(k)}{\lambda(k) + \mathbf{z}^T(k)\Gamma(k-1)\mathbf{z}(k)}\beta(k), \tag{4}$$

$$\Gamma(k) = \Gamma(k-1) - \frac{\Gamma(k-1)\mathbf{z}(k)\mathbf{z}^T(k)\Gamma(k-1)}{\lambda(k) + \mathbf{z}^T(k)\Gamma(k-1)\mathbf{z}(k)}.$$

Here $\hat{\theta}(k) = (\hat{\mathbf{b}}(k), \hat{\mathbf{a}}(k))$, $\hat{\mathbf{b}}^T(k) = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_m)_k$, and $\hat{\mathbf{a}}^T(k) = (\hat{a}_1, \ldots, \hat{a}_m)_k$ are vectors of current estimates of parameters; $\beta(k) = \hat{s}\psi[\alpha(k)]$ with $\alpha(k) = \hat{\varepsilon}(k)/\hat{s}$ for S- and H-algorithms, and $\beta(k) = \hat{s}\hat{\varepsilon}(k)$ for the W-algorithm; $\hat{\varepsilon}(k)/\hat{s} = \{y(k) - \mathbf{z}^T(k)\hat{\theta}(k-1)\}/\hat{s}$ is the same for all the three algorithms, while $\lambda(k) = 1$ for the H-algorithm

$$\lambda(k) = \begin{cases} \{\hat{s}\psi[\alpha(k)]/\hat{\varepsilon}(k)\}^{-1} & \text{for } \hat{\varepsilon}(k) \neq 0, \\ 1 & \text{for } \hat{\varepsilon}(k) = 0, \end{cases} \tag{5}$$

for the W-algorithm; $\lambda(k) = \psi'[\alpha(k)]^{-1}$ for the S-algorithm. Here $\hat{s}$ is the robust estimate of the scale s of residuals. In a case of the S-algorithm the ordinary RLS (4) is modified by substituting the "winsorization" step of the residuals in the first equation and changing the second equation in equations (4). The recursive H-algorithm is obtained only by inserting the "winsorization" step into the first equation of equations (4). The W- algorithm is worked out by inserting different weights in respect to the function $\psi\{\cdot\}$ into the already existing ordinary RLS. In [2] in a case of known parameters of the LQG controller it has been proposed to use $\beta(k) = \hat{s}\phi_{z1}\psi[\alpha(k)/\phi_{z2}]$, and

$$\lambda(k) = \begin{cases} \phi_{z1}\psi[\alpha(k)/\phi_{z2}]/[\alpha(k)/\phi_{z2}] & \text{for } \alpha(k) \neq 0, \\ \phi_{z1} & \text{for } \alpha(k) = 0, \end{cases} \quad (6)$$

respectively. Here $\phi_{z1} = \phi_{z2} = 1$ for Huber's M-estimator; $\phi_{z1} = \phi_z[h(k)]$, for Mallow's, and $\phi_{z1} = \phi_{z2} = \phi_z[h(k)]$, for Shweppe's GM-estimators, respectively, where $\phi_z[h(k)] = \sqrt{1 - h(k)}$ with $h(k) = \mathbf{z}^T(k)\Gamma(k)\mathbf{z}(k)$.

4. Simulation example

A closed-loop system to be simulated is shown in Fig. 1 and described by a linear difference equation of the form [3] $(1 + a_1q^{-1})y(k) = (1 + b_1q^{-1})u(k) + (1 + c_1q^{-1})\xi(k)$, while the controller design equation is $u(k) = e(k) + w_1u(k-1) + w_2u(k-2)$, where $e(k) = r(k) - y(k)$. Here $a_1 = -0.985, b_1 = 2$ and $c_1 = -0.7$. The coefficients of the LQG controller are found according to [3] by the formulas: $w_1 = p + c_1 - a_1$; and $w_2 = (p - a_1)(c - a_1)(b_1 - a_1)^{-1}b$ assuming that time delay is equal to zero and $w_0 = 1$. Then such values of coefficients of the LQG controller are calculated beforehand: $w_1 = 0.1005, w_2 = -0.1016$. The output $y(k), k = 0, 1, 2, \ldots$ of the closed-loop system is observed under the additive noise $v(k)$, $k = 0, 1, 2, \ldots$ containing outliers (see Fig. 2). We calculate estimates of the parameters a_1, b_1, c_1 by processing observations of $\{y(k)\}$ and $\{u(k)\}$ in each current iteration k using the H-algorithm with versions of M-estimator of Huber and Shweppe's GM-estimator, respectively. Afterwards, in each current iteration k the coefficients w_1, w_2 were determined using above mentioned formulas despite which recursive estimation technique of the parameters a_1, b_1, c_1 was used.

10 experiments with different realizations of additive correlated noise $\{v(k)\}$ were carried out in order to investigate more precisely and to compare the accuracy of estimates of the parameter vector θ of the LTI system $G_0(q^{-1};\ \theta)$ simultaneously with the current parameter vector α of the LQG controller $G_R(q^{-1}; \alpha)$, obtained using the H-algorithm with a version of Huber's M-estimator and S-algorithm with version of

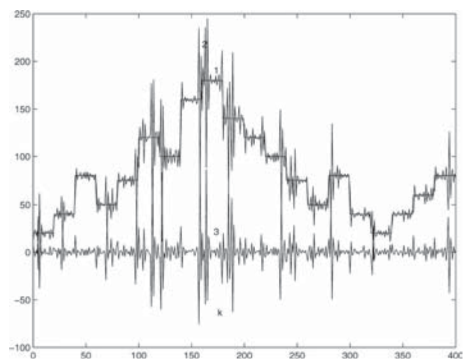

Fig. 2. Signals of a noisy closed-loop system in the presence of outliers in $v(k)$ ($\epsilon = 0.1$): 1) the reference signal $r(k)$, 2) the output $y(k)$ corrupted by an additive noise with outliers in it, 3) the input $u(k)$.

Shweppe's GM-estimator. We have used the Monte Carlo simulation with 10 data sets, each containing 400 input-output observation pairs in the case of additive correlated noise $\{v(k)\}$, having large outliers and corrupting the output $\{y(k)\}$ (see Fig. 2). In each ith experiment the estimates of parameters $a_1 = -0.985, b_1 = 2, c_1 = -0.7$, and $w_1 = 0.1005, w_2 = -0.1016$ have been determined. Table 1 illustrates the values $\bar{b}_1, \bar{a}_1, \bar{c}_1$ of estimates $\hat{b}_1(k), \hat{a}_1(k), \hat{c}_1(k)$, (averaged by 10 experiments), and their confidence intervals

$$\Delta_1 = \pm t_\alpha \frac{\hat{\sigma}_{b_1}}{\sqrt{N}}, \quad \Delta_2 = \pm t_\alpha \frac{\hat{\sigma}_{a_1}}{\sqrt{N}}, \quad \Delta_3 = \pm t_\alpha \frac{\hat{\sigma}_{c_1}}{\sqrt{N}} \quad \forall\, k = \overline{1, 400}. \tag{7}$$

Here $\hat{\sigma}_{b_1}, \hat{\sigma}_{a_1}, \hat{\sigma}_{c_1}$ are estimates of the standard deviations $\sigma_{b_1}, \sigma_{a_1}, \sigma_{c_1}$, respectively; $\alpha = 0.05$ is the significance level; $t_\alpha = 2.26$ is the $100(1-\alpha)\%$ point of Student's distribution with $L-1$ degrees of freedom; $L = 10$ is the number of experiments. Table 2 illustrates the values $\bar{w}_1, \bar{w}_2$ of estimates $\hat{w}_1(k), \hat{w}_2(k)$, (averaged by 10 experiments), and their confidence intervals

$$\Delta_4 = \pm t_\alpha \frac{\hat{\sigma}_{w_1}}{\sqrt{N}}, \quad \Delta_5 = \pm t_\alpha \frac{\hat{\sigma}_{w_2}}{\sqrt{N}} \quad \forall\, k = \overline{1, 400}. \tag{8}$$

Here $\hat{\sigma}_{w_1}, \hat{\sigma}_{w_2}$ are estimates of the standard deviations $\sigma_{w_1}, \sigma_{w_2}$, respectively; Note that in both tables the first line of each k corresponds to the averaged estimates and their confidence intervals which were calculated using the S-algorithm with Shweppe's GM-estimator while the second one – to the same values calculated by the H-algorithm with a version of Huber's M-estimator. The analysis of the estimates, presented in Tables 1, 2, implies that the results obtained by the S-algorithm with Shweppe's GM-estimator corroborate the fact that it is more appreciable than the H-algorithm with a version of Huber's M-estimator because of a higher accuracy of recursive estimates.

Table 1. The averaged estimates of parameters $a_1 = -0.985$, $b_1 = 2$, $c_1 = -0.7$ and their confidence intervals for different k

Observations	*The averaged estimates of parameters*		
k	$\bar{a}_1$	$\bar{b}_1$	$\bar{c}_1$
45	-0.997 ± 0.003 -0.997 ± 0.003	1.081 ± 0.015 1.081 ± 0.015	-0.110 ± 0.008 -0.110 ± 0.008
100	-0.993 ± 0.006 -0.966 ± 0.029	1.371 ± 0.168 2.453 ± 0.074	-0.396 ± 0.089 -0.169 ± 0.251
200	-0.988 ± 0.010 -0.975 ± 0.012	1.689 ± 0.213 2.397 ± 0.098	-0.418 ± 0.054 -0.223 ± 0.241
300	-0.984 ± 0.007 -0.977 ± 0.010	1.925 ± 0.213 2.384 ± 0.094	-0.400 ± 0.004 -0.234 ± 0.228
400	-0.984 ± 0.009 -0.977 ± 0.009	2.075 ± 0.196 2.374 ± 0.092	-0.367 ± 0.003 -0.263 ± 0.216

Table 2. The averaged estimates of parameters $w_1 = 0.1005$, $w_2 = -0.1016$ and their confidence intervals for different k

Observations	*The averaged estimates of parameters*	
k	$\bar{w}_1$	$\bar{w}_2$
45	-0.154 ± 0.003 -0.154 ± 0.003	-0.119 ± 0.002 -0.119 ± 0.002
100	0.115 ± 0.096 -0.194 ± 0.237	-0.097 ± 0.039 -0.178 ± 0.094
200	0.110 ± 0.067 -0.142 ± 0.239	-0.116 ± 0.018 -0.196 ± 0.061
300	0.072 ± 0.100 -0.122 ± 0.224	-0.128 ± 0.020 -0.191 ± 0.058
400	0.028 ± 0.062 -0.102 ± 0.217	-0.144 ± 0.022 -0.186 ± 0.057

References

1. K.J. Åström, Adaptive feedback cotrol, *Proc. of the IEEE*, **75**(2), 185–217 (1987).
2. D.G. Genov, N.R. Atanasov, R. Pupeikis, Robust M- and GM- estimators for closed-loop identification using the direct approach, in: *Proc. of Int. Conf. on Automat. and Informat.*, Sofia (2006), pp. 193–196.
3. M. Halwass, Selbsteinstellende LQG-regelung, *MSR*, **1**, 2–6, **2**, 61–63 (1988).
4. A. Lucas, *Outlier Robust Unit Root Analysis*, Thesis Publishers, Amsterdam (1996), `http://staff.feweb.vu.nl/alucas/thesis/default.htm`
5. J. Novovičova, Recursive computation of M-estimates for the parameters of the linear dynamical system, *Problems of Control and Information Theory*, **16**(1), 19–59 (1987).
6. I.R. Petersen, Minimax LQG control, *Int. J. Appl. Math. Comput. Sci.*, **16**(3), 309–323 (2006).

REZIUMĖ

R. Pupeikis. Apie rekurentinių M- ir GM- įverčių skaičiavimą LQG valdymo sistemose

Straipsnyje vystomas parametrinio LQG (tiesinis kvadratinis Gauso) valdymo sistemų identifikavimo metodas, kai tiesinės pastovių koeficientų sistemos bei LQG reguliatoriaus parametrai esti nežinomi ir turi būti skaičiuojami. Pateikti LQG valdymo sistemos modeliavimo bei jos parametrinio identifikavimo rezultatai.

Raktiniai žodžiai: LQG valdymo sistemos, grįžtamasis ryšys, identifikavimas.