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On limit theorems of Fortet–Kac type

Birutė KRYŽIENĖ (VGTU), Gintautas MISEVIČIUS (VU)
e-mail: gintas.misevicius@mif.vu.lt

Abstract. We get the theorem of large deviations for sums of type $\sum f(T^j t)$ satisfying the conditions weaker than in [5] (see [5, pp. 221–227]).

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Let T be a mapping of $[0,1]$ into itself defined by $T(t) = \{2t\}$ where the braces mean the fractional part of a number. Let the point $t \in [0,1]$ be expressed by

$$t = \frac{\varepsilon_1(t)}{2} + \frac{\varepsilon_2(t)}{2^2} + \cdots + \frac{\varepsilon_k(t)}{2^k} + \cdots .$$

Then the coefficients $\varepsilon_i = \varepsilon_i(t)$, $i = 1, \ldots, k$, are the so-called Rademacher functions, and are independent identically distributed random variables taking values 0 and 1 with equal probabilities.

This expression for irrational numbers is infinite, numbers $t \in (0,1)$ are in a one-to-one correspondence with the sequences $\varepsilon_1(t), \varepsilon_2(t), \ldots$, and every measurable function $f(t)$ can be represented in the form

$$f(t) = f(\varepsilon_1, \varepsilon_2, \ldots).$$

So this function is a random variable, measurable with respect of the σ-algebra, generated by ε_j, $j = 1, 2, \ldots$. The variables $f_k(T^k t) = f_k = f(2^k t) = f(\varepsilon_k, \varepsilon_{k+1}, \ldots)$ form a stationary sequence. For details see [2].

Let $S_n = S_n(t) = \sum_{j=1}^n f(2^j t)$, $f(t)$ being a measurable periodic function with period 1, $\Phi(x)$ denote the normal distribution function.

We suppose that

$$\mathbf{E} f(t) = \int_0^1 f(t)\,\mathrm{d}t = 0, \quad \mathbf{E}|f(t)| \geqslant \delta > 0,$$
$$\mathbf{E}|f(t)|^k = \int_0^1 |f(t)|^k\,\mathrm{d}t \leqslant H_1 H_2^{k-2}(k-2)! \tag{1}$$

and

$$\int_0^1 |f(t+h) - f(t)|^2\,\mathrm{d}t \leqslant H_3 h^\gamma, \tag{2}$$

$k = 3, 4, \ldots, s + 2$, $\sqrt{s} \leqslant \frac{H_0\sqrt{n}}{\rho(n)\ln n} \stackrel{\text{def}}{=} \Delta_n$, $H_0, H_1, H_2, \ldots$ are positive constants, $c_1, c_2, \ldots$ are constants, $\rho(n)$ is a slowly growing function,

$$B_n^2 = \mathbf{D}S_n = \int_0^1 S_n^2(t)\,\mathrm{d}t = \sigma^2 n + c_1. \tag{3}$$

This paper deals with probabilities of large deviations similar to classical results (see [3,4]) for the above defined sums S_n with the condition (2) instead of the following condition in [5]:

$$\int_0^1 \big|f(t+h) - f(t)\big|^p\,\mathrm{d}t \leqslant H_3^p h^\delta p!, \quad p = 2, 3, \ldots .$$

This leads to a narrower zone of validity of large deviations.

THEOREM. *Let $f(t)$ be the above defined Lebesgue measurable function satisfying conditions* (1), (2) *and $\sigma \neq 0$. Then for $1 \leqslant x \leqslant \sqrt{s}$ the following relations of large deviations hold*

$$\frac{\operatorname{meas}_{t\in[0,1]}\{S_n > xB_n\}}{1 - \Phi(x)} = \exp\{L(x)\}\left(1 + \theta_1 f_1(x)\frac{x+1}{\sqrt{s}}\right),$$
$$\frac{\operatorname{meas}_{t\in[0,1]}\{S_n < -xB_n\}}{\Phi(-x)} = \exp\{L(-x)\}\left(1 + \theta_2 f_2(x)\frac{x+1}{\sqrt{s}}\right),$$

where $f_1(x)$, $f_2(x)$ are bounded functions, $|\theta_i| \leqslant 1$, $i = 1, 2$, the power series $L(x) = \sum l_k x^{k+3}$ converges for $|x| < \Delta_n$, the coefficients l_k, $k \geqslant 0$, can be expressed by cumulants, and for $k \leqslant s - 3$ these coefficients coinside with coefficients of the classical Cramér–Petrov series.

Some auxilary statements. Following [1] let us introduce the conditional mean values, i.e., new random variables

$$\zeta_j^{(m)} = [f]_j^{(m)} = [f]_j^{(m)}(t) = \mathbf{E}\big(f\big(T^j t\big)|\varepsilon_{j+1}, \ldots, \varepsilon_{j+m}\big) \tag{4}$$

and

$$\eta_j^{(m)} = \frac{\varepsilon_{j+1}}{2} + \cdots + \frac{\varepsilon_{j+m}}{2^m}. \tag{5}$$

The function $[f]_j^{(k)}$ is measurable by the σ-algebra generated by $\varepsilon_1, \ldots, \varepsilon_k$. It is evident that $\eta_j^{(m)}$ defines a unique set $\varepsilon_{j+1}, \ldots, \varepsilon_{j+m}$, therefore the σ-algebra generated by random variables $\eta_{j_1}^{(m)}, \ldots, \eta_{j_k}^{(m)}$ coincides with the σ-algebra generated by random variables $\varepsilon_{j_1}, \ldots, \varepsilon_{j_k+m}$.

LEMMA 1 (see [5]). *The random variables $\eta_j^{(m)}$, $j = 1, 2, \ldots,$ make the Markov chain. Therefore the random variables $[f]_j^{(m)}$, $j = 1, 2, \ldots,$ are connected into the Markov chain.*

Let $\mathcal{F}_a^b$ be the σ-algebra generated by $\{\eta_k^{(m)},\ k = a, a+1, \ldots, b\}$ and Ω_k be the space of $\eta_k^{(m)}$ values. The coefficients of ergodicity in our case are

$$\alpha_{l,k}^{(m)} = \begin{cases} 0 & \text{for } k - l \leqslant m, \\ 1 & \text{for } k - l \geqslant m+1, \end{cases}$$

since $\varepsilon_1, \varepsilon_2, \ldots$ are independent identically distributed random variables.

Let us consider the “modified” sum

$$S_n' = \sum_{j=1}^{n} [f]_j^{(m)}.$$

LEMMA 2. *Under the conditions* (1) *and* (2)*, the following relations are valid:*

$$B_n^2 = \mathbf{D}S_n = \sigma^2 n + \theta_n, \quad |\theta_n| < c_1, \tag{6}$$

$$B_n'^2 = \mathbf{D}S_n' = \sigma_m^2 n + \theta_m, \quad |\theta_m| < c_2, \tag{7}$$

and

$$|\sigma - \sigma_m| \leqslant \frac{c_3}{(\sqrt{2})^{m\gamma}}, \tag{8}$$

where

$$\sigma^2 = \int_0^1 f^2(t)\,\mathrm{d}t + 2\sum_{k=1}^{\infty} f(t) f\left(2^k t\right) \mathrm{d}t \neq 0. \tag{9}$$

Proof. Following the reasoning of I.A. Ibragimov [1] we get the estimation

$$\mathbf{E}^{1/2}\left|\zeta_j - \zeta^{(m)}\right|^2 \leqslant c_4 2^{-m\gamma}, \tag{10}$$

and using the reasoning in §2 and §3 of Chapter XVIII in [2], we find

$$\mathbf{D}S_n = n\mathbf{E}\zeta_1^2 + \sum_{j=1}^{n} (n-j)\mathbf{E}\zeta_1\zeta_j$$

with the estimation

$$|\mathbf{E}\zeta_1\zeta_{k+1}| \leqslant 4 \cdot 2^{-\frac{\gamma}{2}k} c_\gamma \mathbf{E}^{1/2}\zeta_1^2. \tag{11}$$

An analogous estimation takes place for $\mathbf{D}S_n'$ and $|\mathbf{E}\zeta_1^{(m)}\zeta_j^{(m)}|$.

Thus we get

$$\mathbf{D}S_n = n\left(\mathbf{E}\zeta_1^2 + \sum_{j=1}^{\infty}\mathbf{E}\zeta_1\zeta_j\right) + n\sum_{j=n+1}^{\infty}\mathbf{E}\zeta_1\zeta_j - \sum_{j=1}^{n}\mathbf{E}\zeta_1\zeta_j$$

and the second and third members do not exceed

$$\left|\frac{4\cdot 2^{\frac{\gamma}{2}} c_\gamma \mathbf{E}^{1/2}\zeta_1^2}{2^{\frac{\gamma}{2}n}}\right| \leqslant c_5 \quad \text{and} \quad c_6 = \sum_{k=1}^{\infty} \frac{4\cdot 2^{\frac{\gamma}{2}} c_\gamma \mathbf{E}\zeta_1^2}{2^{k\frac{\gamma}{2}-1}} < \infty.$$

Thus we obtain (6), and in a similar manner one can get (7).

Further we get by the Minkovsky inequality

$$\left|\mathbf{E}\zeta_1^2 - \mathbf{E}\big(\zeta_1^{(m)}\big)^2\right| \leqslant 8c_\gamma \mathbf{E}\zeta_1^2 2^{-\frac{\gamma}{2}m},$$

and

$$\left|\mathbf{E}\zeta_1\zeta_{k+1} - \mathbf{E}\zeta_1^{(m)}\zeta_{k+1}^{(m)}\right| \leqslant \min\left(\frac{8\cdot 2^{\frac{\gamma}{2}} c_\gamma \mathbf{E}^{1/2}\zeta_1^2}{k^{\frac{\gamma}{2}}}, \frac{8\mathbf{E}\zeta_1^2 c_\gamma}{2^{\frac{\gamma}{2}m}}\right).$$

So we obtain

$$|\sigma - \sigma_m| \leqslant \left|\mathbf{E}\zeta_1^2 - \mathbf{E}\zeta_1^{(m)\,2}\right| + \sum_{j=1}^{m}\left|\mathbf{E}\zeta_1\zeta_j - \mathbf{E}\zeta_1^{(m)}\zeta_j^{(m)}\right|$$

$$+ \sum_{j=m+1}^{\infty}\left|\mathbf{E}\zeta_1\zeta_j - \mathbf{E}\zeta_1^{(m)}\zeta_j^{(m)}\right| \leqslant \frac{c_3}{2^{m\frac{\gamma}{2}-1}},$$

where $c_3 = 9(2^{\frac{\gamma}{2}} c_\gamma \mathbf{E}^{1/2}\zeta_1^2 + c_\gamma \mathbf{E}\zeta_1^2)$.

Lemma is proved.

LEMMA 3 (see [3]). *Let a random variable ζ with $\mathbf{E}\zeta = 0$ and $\mathbf{E}\zeta^2 = 1$ satisfy the inequality*

$$\Gamma_k(\zeta) = (k-2)!/\Delta^{k-2}, \quad k = 3, 4, \ldots, s+2,$$

where s is even and $s \leqslant 2\Delta^2$.

Then for x, $0 \leqslant x \leqslant \sqrt{s}/3\sqrt{e}$, the following relation of large deviations is valid:

$$\frac{1 - F_\zeta(x)}{1 - \Phi(x)} = \exp\big\{\tilde{L}(x)\big\}\left(1 + \theta_s f_1(x)\frac{x+1}{\sqrt{s}}\right), \quad x > 0,$$

and a similar one for negative x.

The power series

$$\tilde{L}(x) = \sum_{k=0}^{\infty} \tilde{l}_k x^{k+3}$$

is convergent for $|x| < \sqrt{2}\Delta/3\sqrt{e}$ and $f_1(x)$ is a bounded function.

Proof of Theorem. Taking in (8) $m = \frac{\rho(n)\ln n}{\gamma}$ we get

$$|\sigma - \sigma_m| \leqslant \frac{c_7}{n^{\rho(n)}}.$$

By the conditions (1) and (2), using the Schwartz inequality we get

$$\int_0^1 \left|f(t+h)-f(t)\right|\mathrm{d}t \leqslant \left(\int_0^1 \left|f(t+h)-f(t)\right|^2\mathrm{d}t\right)^{1/2}$$
$$\times\left(\int_0^1 f(t+h)-f(t)\,\mathrm{d}t\right)^{1/2} \leqslant h^{\alpha}2^k H_5^k k!,$$

and we deduce, using the Hölder and Jensen inequalities, that

$$\left|\mathbf{E}\left|f\left(T^j t\right)\right|^k - \mathbf{E}\left|\zeta_j^{(m)}\right|^k\right| \leqslant \frac{c_8}{n^{\rho(n)}},$$
$$\mathbf{E}^{1/k}\left|[f_j]^{(m)}\right|^k \leqslant \mathbf{E}^{1/k}\left|f(t)\right|^k + \frac{H_1}{n^{\rho(n)}},$$
$$\mathbf{E}\left|[f]_j^{(m)}\right| \leqslant \tilde{H}_{12}H_{13}^{k-2}k!, \quad j=1,2,\ldots. \tag{12}$$

Using (12) it is enough to get the estimate of cumulants of S_n'

$$\left|\Gamma_k\left(\frac{S_n'}{\sigma_m\sqrt{n}}\right)\right| \leqslant k!H_{14}\left(\frac{H_{15}\rho(n)\ln n}{\sigma_m\sqrt{n}}\right)^{k-2}.$$

Further conclusions coinside with those of [4], and the difference

$$\frac{S_n}{\sigma_m\sqrt{n}} - \frac{S_n'}{\sigma_m\sqrt{n}}$$

can be evaluated following [5] word by word.

Theorem is proved.

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REZIUMĖ

B. Kryžienė, G. Misevičius. Ribinės teoremos Forte–Kaco tipo sumoms

Darbe įrodyta didžiųjų nuokrypių ribinė teorema sumoms $\sum f(T^j t)$ su silpnesniais apribojimais periodinei funkcijai $f(t)$ negu ankstesniuose autorių darbuose (žr. [5]).

Raktiniai žodžiai: dvimatis toras, didieji nuokrypiai, semiinvariantai.