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A property of the uniform distribution modulo 1

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Abstract. An interesting relationship between Farey fractions and the uniform distribution modulo 1 is discovered.

Keywords: Farey fractions, uniform distribution modulo 1.

1. The results

Let α be an irrational number in $(0;1)$ and, for $i \geqslant 0$, s_i be the fractional part of $i\alpha$: $s_i = i\alpha - \lfloor i\alpha \rfloor$ with $\lfloor\cdot\rfloor$ standing for the integer part.

Let $\mathcal{S}_n$ denote the set of all intervals of the form $(s_{(k)}; s_{(k+1)})$; here $0 \leqslant k \leqslant n$, $s_{(0)}, \ldots, s_{(n)}$ are the numbers $s_0, \ldots, s_n$ sorted in ascending order and $s_{(n+1)} = 1$. Let $I_0 = (a_0; b_0) \in \bigcup_{i \geqslant 1} \mathcal{S}_i$. For $n \geqslant 1$ define recursively

$$i_n = \min\{i \mid s_i \in I_{n-1}\}, \tag{1.1}$$

$$I_n = (a_n; b_n) = \begin{cases} (a_{n-1}; s_{i_n}) & \text{if } s_{i_n} > (a_{n-1} + b_{n-1})/2; \\ (s_{i_n}; b_{n-1}) & \text{otherwise.} \end{cases} \tag{1.2}$$

In other words, s_{i_n} is the first s_i in I_{n-1} and I_n is the greatest of two connected components of $I_n \backslash \{s_{i_n}\}$. Let

$$t_n = \frac{s_{i_n} - a_{n-1}}{b_{n-1} - a_{n-1}}. \tag{1.3}$$

The main result of the paper is the following theorem.

THEOREM 1. *For all α there exists an n such that $t_n \in (1/3; 2/3)$.*

Theorem 1 follows from two others results that are formulated below. But first we introduce some additional notions.

For $n \geqslant 1$, let $F_n = \{(i,j) \mid 0 \leqslant j \leqslant i \leqslant n,\ \langle i,j\rangle = 1\}$; here $\langle i,j\rangle$ stands for the greatest common divisor of i and j. If $(i,j) \in F_n$, the number j/i is called *a Farey fraction of order n*. Let $\mathcal{F}_n$ denote the set of all intervals of the form $(j_{k-1}/i_{k-1}; j_k/i_k)$; here $k = 1, \ldots, K$ and $j_0/i_0, \ldots, j_K/i_K$ are all Farey fractions of order n sorted in ascending order. For example,

$$\mathcal{F}_4 = \{(0;1/4), (1/4;1/3), (1/3;1/2), (1/2;2/3), (2/3;3/4), (3/4;1)\}.$$

In what follows a formula $(j/i;\, j'/i') \in \mathcal{F}_n$ will implicitly mean that $\langle i, j\rangle = \langle i', j'\rangle = 1$.

Let $\mathcal{F} = \bigcup_{n\geqslant 1} \mathcal{F}_n$. It is known from the theory of Farey fractions [1, Chapter III] that if $I = (j/i;\, j'/i') \in \mathcal{F}$ and $i'' = i + i'$, $j'' = j + j'$, then the fraction j''/i'' is also irreducible and lies in I. It is called *the mediant of the fractions j/i and j'/i'*. The mediant is the unique number a with the following property: $(j/i;\, a), (a;\, j'/i') \in \mathcal{F}$. This implies that if $(j/i;\, j'/i') \in \mathcal{F}$ then

$$(j/i;\, j'/i') \in \mathcal{F}_n \iff i \vee i' \leqslant n < i + i'$$

(here and in the sequel $a \vee b = \max(a, b)$).

Finally, denote $s'_n = s_n$ for $n \geqslant 1$ and $s'_0 = 1$.

THEOREM 2. *If $\alpha \in (j/i;\, j'/i') \in \mathcal{F}_n$ then:*
(i) *$s_i = i\alpha - j$, $(0; s_i) \in \mathcal{S}_n$, $1 - s'_i = j' - i'\alpha$ and $(s_{i'}; 1) \in \mathcal{S}_n$;*
(ii) *$s_n - s_{n-i} = s_i$, $(s_{n-i}; s_n) \in \mathcal{S}_n$, $s'_{n-i'} - s_n = 1 - s_{i'}$ and $(s_n; s'_{n-i'}) \in \mathcal{S}_n$.*

THEOREM 3. *Let $(s_k; s'_l) \in \mathcal{S}_{k\vee l}$, $\alpha \in (j/i;\, j'/i') \in \mathcal{F}_{k\vee l}$ and r be the first index with $s_r \in (s_k; s'_l)$.*

(i) *If $k < l$ then $r = p + k$ and $s_r - s_k = s_p = p\alpha - q$; here $p = mi + i'$, $q = mj + j'$ and m is defined by*

$$\alpha \in \Big(\frac{mj + j'}{mi + i'};\, \frac{(m-1)j + j'}{(m-1)i + i'}\Big). \tag{1.4}$$

(ii) *If $k > l$ then $r = p + l$ and $s'_l - s_r = 1 - s_p = q - p\alpha$; here $p = i + mi'$, $q = j + mj'$ and m is defined by*

$$\alpha \in \Big(\frac{j + (m-1)j'}{i + (m-1)i'};\, \frac{j + mj'}{i + mi'}\Big). \tag{1.5}$$

2. Proofs

Proof of Theorem 2. First prove the statements concerning s_i and s_{n-i}.

(i) Inequalities

$$\frac{j}{i} < \alpha < \frac{j'}{i'} \leqslant \frac{j+1}{i}$$

imply $j < i\alpha < j + 1$. Hence $s_i = i\alpha - j$.

Suppose $(0; s_i) \notin \mathcal{S}_n$ and find $l \leqslant k \leqslant n$ such that $0 < s_k = k\alpha - l < s_i$. Without loss of generality we can assume that $\langle k, l\rangle = 1$; then $l/k \neq j/i$. Inequality $s_k > 0$ implies $l/k < \alpha$. Since $l/k \notin (j/i;\, j'/i')$, this yields

$$\frac{l}{k} < \frac{j}{i}. \tag{2.1}$$

Inequality $s_k < s_i$ implies $(i - k)\alpha > j - l$. If $i > k$ then $\alpha > \frac{j-l}{i-k}$ and therefore

$$\frac{j - l}{i - k} \leqslant \frac{j}{i}.$$

This contradicts to (2.1), because j/i is the mediant of l/k and $(j-l)/(i-k)$. If $i<k$ then $\alpha<\frac{l-j}{k-i}$ and therefore

$$\frac{j}{i}<\frac{l-j}{k-i}.$$

This contradicts to (2.1), because l/k is the mediant of j/i and $(l-j)/(k-i)$. Hence $(0;s_i)\in\mathcal{S}_n$.

(ii) If $n=1$ then $i=i'=1$, $j=0$ and $j'=1$; therefore statement (ii) reduces to $s_1-s_0=s_1$, $(0;s_1)\in\mathcal{S}_1$. Obviously, it holds true.

Let $n\geqslant 2$. If $i=n$, then equality $s_n-s_{n-i}=s_i$ is trivial and $(s_{n-i};s_n)\in\mathcal{S}_n$ follows from (i). Now let $i\leqslant n-1$. Then $\alpha\in(j/i;\,j'/i')\subset(j/i;*)\in\mathcal{F}_{n-1}$ and, by the inductive assumption, $s_{n-1}-s_{n-1-i}=s_i$. Let $s_{n-1}=(n-1)\alpha-m$; then, by (i), $s_{n-1-i}=s_{n-1}-s_i=(n-1-i)\alpha-(m-j)$.

Now there are two possibilities: $s_{n-1}+\alpha<1$ and $s_{n-1}+\alpha>1$. In the first case *a fortiori* $s_{n-1-i}+\alpha<1$; therefore $s_{n-i}=s_{n-1-i}+\alpha$ and $s_n=s_{n-1}+\alpha$. In the second case $n\alpha-m>1$, i.e., $\alpha>\frac{m+1}{n}$. This implies

$$\frac{m+1}{n}\leqslant\frac{j}{i}.$$

But $(m+1)/n$ is the mediant of $(m-j+1)/(n-i)$ and j/i. Therefore $(m-j+1)/(n-i)\leqslant j/i<\alpha$, i.e., $(n-i)\alpha-(m-j+1)>0$, i.e., $s_{n-1-i}+\alpha>1$. Hence $s_{n-i}=s_{n-1-i}+\alpha-1$ and $s_n=s_{n-1}+\alpha-1$. In both cases

$$s_n-s_{n-i}=s_{n-1}-s_{n-1-i}=s_i.$$

Suppose $(s_{n-i};s_n)\notin\mathcal{S}_n$ and find $k\leqslant n$ such that $s_{n-i}<s_k<s_n$. Then

$$0\leqslant s_{n-1-i}<s_k-\beta<s_{n-1}<1;$$

here $\beta=\alpha$ in the first case and $\beta=\alpha-1$ in the second. Therefore $s_k-\beta=s_{k-1}$ and $s_{k-1}\in(s_{n-1-i};s_{n-1})$. This contradicts to the inductive assumption $(s_{n-1-i};s_{n-1})\in\mathcal{S}_{n-1}$. Hence $(s_{n-i};s_n)\in\mathcal{S}_n$.

Statements concerning $s_{i'}$ and $s'_{n-i'}$ can be deduced from those about s_i and s_{n-i}. Since

$$1-\alpha\in\left(\frac{i'-j'}{i'};\frac{i-j}{i}\right)\in\mathcal{F}_n,$$

we can apply them to get some properties of $\tilde{s}_n=n(1-\alpha)-\lfloor n(1-\alpha)\rfloor$. But first find the relation between $\tilde{s}_n$ and s_n.

Obviously, $\tilde{s}_0=0=1-s'_0$. If $n>0$ and $s_n=n\alpha-m$ then

$$m<n\alpha<m+1;$$
$$n-m-1<n-n\alpha<n-m;$$
$$\lfloor n(1-\alpha)\rfloor=n-m-1;$$
$$\tilde{s}_n=n(1-\alpha)-n+m+1=1-n\alpha+m=1-s_n.$$

Therefore $\tilde{s}_n = 1 - s'_n$ for all n.

Now we get:

$1 - s_{i'} = i'(1-\alpha) - (i' - j') = j' - i'\alpha$;

$1 - s_k \notin (0; 1 - s_{i'})$ for $k \leqslant n$, i.e., $s_k \notin (s_{i'}; 1)$ for $k \leqslant n$, i.e., $(s_{i'}; 1) \in \mathcal{S}_n$;

$1 - s_n - 1 + s'_{n-i'} = 1 - s_{i'}$, i.e., $s'_{n-i'} - s_n = 1 - s_{i'}$;

$1 - s_k \notin (1 - s'_{n-i'}; 1 - s_n)$ for $k \leqslant n$, i.e., $s_k \notin (s_n; s'_{n-i'})$ for $k \leqslant n$, i.e., $(s_n; s'_{n-i'}) \in \mathcal{S}_n$.

Proof of Theorem 3. Let $k < l$; then by Theorem 2 $k = l - i$. If $l < n < mi + i'$ then $\alpha \in (j/i; *) \in \mathcal{F}_n$ and by Theorem 2 $(s_{n-i}; s_n) \in \mathcal{S}_n$. Since $n - i > l - i = k$, $s_{n-i} \neq s_k$ and therefore $n \neq r$ (because $(s_k; s_r) \in \mathcal{S}_r$).

If $mi + i' \leqslant n \leqslant mi + i' + k$ then the interval in (1.4) lies in $\mathcal{F}_n$, because $i + i' > l$ implies

$$mi + i' + (m-1)i + i' \geqslant mi + 2i' > mi + i' + l - i = mi + i' + k \geqslant n.$$

By Theorem 2, $(s_{n-mi-i'}; s_n) \in \mathcal{S}_n$. If $n < mi + i' + k$ then $s_{n-mi-i'} \neq s_k$ and therefore $n \neq r$. If $n = mi + i' + k$ then $s_{n-mi-i'} = s_k$. Hence $r = mi + i' + k$.

Theorem 2 also yields $s_r - s_k = s_p = p\alpha - q$ with $q = mj + j'$.

The case $k > l$ is considered analogously.

Proof of Theorem 1. Suppose the contrary, $t_n \notin (1/3; 2/3)$ for all n. Let $I_0 = (s_k; s_l) \in \mathcal{S}_l$ (so $k < l$; the case $k > l$ will be considered lately). Let $\alpha \in (j/i; j'/i') \in \mathcal{F}_l$ and m be defined by (1.4). Denote $p = mi + i'$ and $q = mj + j'$.

Suppose firstly $t_1 < 1/3$ and prove that, for all $n \geqslant 1$,

$$t_n < 1/3, \quad i_n = np + k, \quad s_l - s_{i_n} = (nq - j) - (np - i)\alpha \tag{2.2}$$

and

$$\alpha \in \left(\frac{q}{p}; \frac{q - \frac{j}{n+2}}{p - \frac{i}{n+2}}\right). \tag{2.3}$$

By Theorem 2, $k = l - i$ and $s_l - s_k = s_i = i\alpha - j$. By Theorem 3, $i_1 = p + k$ and $s_{i_1} - s_k = s_p = p\alpha - q$; therefore $s_l - s_{i_1} = (q - j) - (p - i)\alpha$. Hence

$$t_1 = \frac{s_{i_1} - s_k}{s_l - s_k} = \frac{s_p}{s_i}$$

and $t_1 < 1/3$ yields $s_p - \frac{1}{3}s_1 < 0$, i.e., $(p - \frac{i}{3})\alpha - (q - \frac{j}{3}) < 0$. Therefore (2.2)–(2.3) hold true for $n = 1$.

Now let $n \geqslant 2$. By the inductive assumption, $t_1, \ldots, t_{n-1} < 1/2$; therefore i_n is the first index with $s_{i_n} \in (s_{i_{n-1}}; s_l)$. Again by the inductive assumption, $i_{n-1} = (n-1)p + k$ and

$$\alpha \in \left(\frac{q}{p}; \frac{q - \frac{j}{n+1}}{p - \frac{i}{n+1}}\right) \subset \left(\frac{q}{p}; \frac{q - \frac{j}{n-1}}{p - \frac{i}{n-1}}\right) = \left(\frac{q}{p}; \frac{(n-1)q - j}{(n-1)p - i}\right).$$

The interval in the right-hand side lies in $\mathcal{F}_{i_{n-1}}$, because $p \geqslant i + i' > l = k + i$ and $-i < k < p - i$ implies

$$(n-1)p - i < i_{n-1} < (n-1)p - i + (p - i).$$

Therefore, by Theorem 3, $i_n = p + m\big[(n-1)p - i\big] + l$; here m is defined by

$$\alpha \in \left(\frac{q + (m-1)[(n-1)q - j]}{p + (m-1)[(n-1)p - i]}; \frac{q + m[(n-1)q - j]}{p + m[(n-1)p - i]}\right).$$

By the inductive assumption, $m = 1$, because then the left end of this interval is $q/p < \alpha$ and the right end

$$\frac{nq - j}{np - i} = \frac{q - \frac{j}{n}}{p - \frac{i}{n}} > \frac{q - \frac{j}{n+1}}{p - \frac{i}{n+1}} > \alpha.$$

Hence $i_n = np - i + l = np + k$. Moreover, $s_l - s_{i_n} = 1 - s_{np-i} = (nq - j) - (np - i)\alpha$.

Now

$$1 - t_n = \frac{s_l - s_{i_n}}{s_l - s_{i_{n-1}}} = \frac{(nq - j) - (np - i)\alpha}{[(n-1)q - j] - [(n-1)p - i]\alpha} > \frac{1}{2},$$

because

$$\alpha < \frac{q - \frac{j}{n+1}}{p - \frac{i}{n+1}} = \frac{(n+1)q - j}{(n+1)p - i}$$

implies $[(n+1)p - i]\alpha < (n+1)q - j$ and therefore

$$2(nq - j) - 2(np - i)\alpha > (n-1)q - j - [(n-1)p - i]\alpha.$$

Hence $t_n < 1/2$ and therefore $t_n < 1/3$, because $t_n \notin (1/3; 2/3)$. But then

$$1 - t_n = \frac{(nq - j) - (np - i)\alpha}{(n-1)q - j - [(n-1)p - i]\alpha} > \frac{2}{3},$$

i.e.,

$$(n+2)q - j > [(n+2)p - i]\alpha;$$

$$\alpha < \frac{(n+2)q - j}{(n+2)p - i} = \frac{q - \frac{j}{n+2}}{p - \frac{i}{n+2}}.$$

Relations (2.2)–(2.3) are thus proved. But (2.3) can not hold for all n. Therefore assumption $t_1 < 1/3$ leads to contradiction.

If $t_1 > 2/3$ then similarly as above is proved that for all $n \geqslant 1$

$$t_n > 2/3, \quad i_n = n(p - i) + l, \quad s_{i_n} - s_k = [n(p - i) + i]\alpha - [n(q - j) + j]$$

and

$$\alpha \in \Big(\frac{q - j + \frac{j}{n+2}}{p - i + \frac{i}{n+2}}; \frac{q - j}{p - i}\Big). \tag{2.4}$$

But (2.4) can not hold for all n. Therefore $t_1 > 2/3$ also leads to contradiction.

The theorem is thus proved in the case $I_0 = (s_k; s_l) \in \mathcal{F}_l$. Now let $k > l$, i.e., $I_0 = (s_k; s_l') \in \mathcal{F}_k$. Denote $\tilde{s}_n = n(1-\alpha) - \lfloor n(1-\alpha)\rfloor$, $\tilde{I}_0 = (\tilde{a}_0; \tilde{b}_0) = 1 - I_0$, and, for $n \geqslant 1$, define $\tilde{i}_n$, $\tilde{I}_n = (\tilde{a}_n; \tilde{b}_n)$ and $\tilde{t}_n$ by (1.1)–(1.3) but with $\tilde{s}_n$ and $\tilde{I}_0$ instead of s_n and I_0.

In the proof of Theorem 2 we got $\tilde{s}_n = 1 - s_n'$. Therefore if some $\tilde{s}_n \in \tilde{I}_0$ then $s_n \in I_0$ and *vice versa*. Hence $\tilde{i}_1 = i_1$. Moreover,

$$\tilde{t}_1 = \frac{\tilde{s}_{i_1} - \tilde{a}_0}{\tilde{b}_0 - \tilde{a}_0} = \frac{(1 - s_{i_1}) - (1 - b_0)}{(1 - a_0) - (1 - b_0)} = \frac{b_0 - s_{i_1}}{b_0 - a_0} = 1 - t_1.$$

If $\tilde{t}_1 < 1/2$ then $t_1 > 1/2$ and therefore $\tilde{I}_1 = (\tilde{s}_{i_1}; \tilde{b}_0) = (1 - s_{i-1}; 1 - a_0) = 1 - I_1$. If $\tilde{t}_1 > 1/2$, equality $\tilde{I}_1 = 1 - I_1$ is proved analogously.

Now repeat these arguments to get $\tilde{i}_n = i_n$, $\tilde{I}_n = 1 - I_n$ and $\tilde{t}_n = 1 - t_n$ for all n. Since $\tilde{I}_0 = (\tilde{s}_l; \tilde{s}_k) \in \mathcal{F}_k$, we can apply the proven part of the theorem and get $\tilde{t}_n \in (1/3; 2/3)$ for some n. Then t_n also lies in $(1/3; 2/3)$.

References

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REZIUMĖ

V. Kazakevičius. Viena tolygaus pasiskirstymo moduliu 1 savybė

Rastas įdomus ryšys tarp Farey trupmenų ir tolygaus pasiskirstymo moduliu 1.

Raktiniai žodžiai: Farey trupmenos, tolygus pasiskirstymas moduliu 1.