On the rate of convergence of L_p norms in the CLT for Poisson random sum[*]

Jonas Kazys SUNKLODAS

Institute of Mathematics and Informatics
Akademijos 4, LT-08663 Vilnius, Lithuania
Vilnius Gediminas Technical University
Saulėtekio 11, 10223 Vilnius, Lithuania
e-mail: sunkl@ktl.mii.lt

Abstract. In the paper, we present the upper bound of L_p norm $\Delta_{\lambda,p}$ of the order $\lambda^{-\delta/2}$ for all $1 \leqslant p \leqslant \infty$, in the central limit theorem for a standardized random sum $(S_{N_\lambda} - \mathbb{E}S_{N_\lambda})/\sqrt{\mathbb{D}S_{N_\lambda}}$, where $S_{N_\lambda} = X_1 + \cdots + X_{N_\lambda}$ is the random sum of independent identically distributed random variables $X, X_1, X_2, \ldots$ with $\beta_{2+\delta} = \mathbb{E}|X|^{2+\delta} < \infty$ where $0 < \delta \leqslant 1$, N_λ is a random variable distributed by the Poisson distribution with the parameter $\lambda > 0$, and N_λ is independent of $X_1, X_2, \ldots$.

Keywords: central limit theorem, L_p norms, Poisson random sum.

1. Introduction

Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables (r.v.'s) with $\mu = \mathbb{E}X$, $\alpha_2 = \mathbb{E}X^2$, $\sigma^2 = \mathbb{D}X$, and

$$\beta_{2+\delta} = \mathbb{E}|X|^{2+\delta} < \infty, \quad 0 < \delta \leqslant 1. \tag{1}$$

Let N_λ be a r.v. distributed by the Poisson distribution with the parameter $\lambda > 0$ (for short, $N_\lambda \sim \mathcal{P}(\lambda)$), i.e.,

$$\mathbb{P}\{N_\lambda = k\} = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad k = 0, 1, \ldots. \tag{2}$$

Moreover, assume that the r.v. N_λ is independent of the $X_1, X_2, \ldots$, and consider the so-called Poisson random sum

$$S_{N_\lambda} = X_1 + \cdots + X_{N_\lambda} \quad \left(\sum_{i=1}^{0} = 0\right).$$

Denote

$$Z_{N_\lambda} = \frac{S_{N_\lambda} - \mathbb{E}S_{N_\lambda}}{\sqrt{\mathbb{D}S_{N_\lambda}}} = \frac{S_{N_\lambda} - \lambda\mu}{\sqrt{\lambda\alpha_2}}, \qquad \varkappa^{2+\delta} = \frac{\beta_{2+\delta}}{(\mu^2 + \sigma^2)^{(2+\delta)/2}},$$

$$\Delta_\lambda(x) = \mathbb{P}\{Z_{N_\lambda} < x\} - \Phi(x), \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} \mathrm{e}^{-u^2/2} \, \mathrm{d}u,$$

$$\Delta_{\lambda,p} = \begin{cases} \left(\int_{-\infty}^{\infty} |\Delta_\lambda(x)|^p \, \mathrm{d}x \right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \sup_{x \in \mathbb{R}} |\Delta_\lambda(x)| & \text{if } p = \infty. \end{cases}$$

Here and in what follows $\mathbb{R}$ is a real line.

We are interested in the rate of convergence of the L_p norm $\Delta_{\lambda,p}$ for all $1 \leqslant p \leqslant \infty$. First of all, note that there is a rich literature on normal approximations for random sums of independent sequences of r.v.'s, see, for example, papers [4,9], books [7,11], and the references therein. In most of these works the upper bounds of the uniform distance $\Delta_{\lambda,\infty}$ are investigated. The obtained bounds contain, as a rule, several terms, outlined by the distributions of summands $X_1, X_2, \ldots$ (not necessarily identically distributed r.v.'s), and the random numbers of summands N are also not necessarily Poissonian. These general bounds are provided with some loss of accuracy for concrete distribution of the random number of summands N.

Recall that Robbins (1948) [16] has proved that, if a distribution of a r.v. N depends on the parameter λ, the r.v. N has positive finite variance, i.e., $0 < \mathbb{D}N < \infty$, and

$$\lim_{\lambda \to \infty} \mathbb{P}\left\{ \frac{N - \mathbb{E}N}{\sqrt{\mathbb{D}N}} < x \right\} = \Phi(x), \tag{3}$$

then for a random sum $S_N = X_1 + \cdots + X_N$, where $X_1, X_2, \ldots$ are independent identically distributed r.v.'s and N is independent of $X_1, X_2, \ldots$, the following equality is valid

$$\lim_{\lambda \to \infty} \mathbb{P}\left\{ \frac{S_N - \mathbb{E}S_N}{\sqrt{\mathbb{D}S_N}} < x \right\} = \Phi(x). \tag{4}$$

In [20], we have obtained the upper estimates of L_p norms for all $1 \leqslant p \leqslant \infty$ of the order $(\mathbb{D}N)^{-1/2}$ in the limit (3) as N is either a Poisson r.v. or gamma one.

In this paper, we have obtained the upper estimate of the L_p norm $\Delta_{\lambda,p}$ for all $1 \leqslant p \leqslant \infty$ (see Theorem 1). The obtained constants are not the best possible, but that was not the main author's aim.

The uniform estimate (5) of $\Delta_{\lambda,\infty}$ in the case $\delta = 1$ was first presented in [15], and the proof independently was given in [3] and [1]. It has been proved independently in [14] and [10] that the constant $C_\infty(1)$ in (5) in this particular case ($p = \infty$, $\delta = 1$) is the same as in the Berry–Esséen inequality for the sum of a non-random number of independent identically distributed summands ($C_\infty(1) \leqslant 0.7655$ [17]).

As in [20], to obtain the upper estimates of the norm $\Delta_{\lambda,\infty}$ (uniform metric) and the L_1 norm $\Delta_{\lambda,1}$, we have formed linear differential equations from the characteristic function of the standardized r.v. $Z_{N_\lambda} = \frac{S_{N_\lambda} - \lambda\mu}{\sqrt{\lambda\alpha_2}}$ by virtue of which we succeeded in getting proper estimates of differences: between the characteristic function and the normal one, and between their derivatives as well. The proof of the estimate of the L_p norm is elementary.

The papers, for example, [5] and [19], in which the rate of convergence of L_p norms in the central limit theorem is investigated for sums of independent random variables, are close to that of ours.

The obtained results are produced under the influence of Stein's method and the papers of Stein [18] and Tikhomirov [21].

2. Main and auxiliary results

Now, we formulate the main result.

THEOREM 1. *Let* $X, X_1, X_2, \ldots$ *be independent identically distributed r.v.'s with* $\mu = \mathbb{E}X$, $\alpha_2 = \mathbb{E}X^2$, $\sigma^2 = \mathbb{D}X$, $\beta_{2+\delta} = \mathbb{E}|X|^{2+\delta} < \infty$, *for some* $0 < \delta \leqslant 1$, *and the r.v.* $N_\lambda \sim \mathcal{P}(\lambda)$ *with the parameter* $\lambda > 0$ *be independent of* $X_1, X_2, \ldots$. *Then for all* $1 \leqslant p \leqslant \infty$

$$\Delta_{\lambda, p} \leqslant C_p(\delta) \frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}, \tag{5}$$

where the constant $C_p(\delta)$ *depends only on* δ *and* p. *Moreover,* $C_\infty(1) = \frac{2}{3\sqrt{\pi}} + \frac{8}{\pi}\sqrt{\frac{2}{\pi}} < 2.41$, *and* $C_p(1) < 20.87$ *for all* $1 \leqslant p \leqslant \infty$.

Recall that $\mathbb{E}N_\lambda = \mathbb{D}N_\lambda = \lambda$ for the r.v. $N_\lambda \sim \mathcal{P}(\lambda)$.

Denote the characteristic function of the standardized r.v. $Z_{N_\lambda} = \frac{S_{N_\lambda} - \lambda\mu}{\sqrt{\lambda\alpha_2}}$ by $f_\lambda(t) = \mathbb{E}e^{itZ_{N_\lambda}}$, and the derivative of the characteristic function $f_\lambda(t)$ with respect to t by $f_\lambda'(t)$.

To prove Theorem 1, we use an auxiliary result, Lemma 1, on the behaviour of the functions $f_\lambda(t)$ and $f_\lambda'(t)$.

LEMMA 1. *Let the conditions of Theorem 1 be satisfied. Then the characteristic function* $f_\lambda(t)$ *satisfies the following homogeneous linear differential equation for all* $t \in \mathbb{R}$:

$$f_\lambda'(t) = i\sqrt{\frac{\lambda}{\alpha_2}}\left(\mathbb{E}Xe^{i\frac{t}{\sqrt{\lambda\alpha_2}}X} - \mu\right)f_\lambda(t). \tag{6}$$

Moreover, for all $|t| \leqslant C_0(\delta)\frac{\sqrt{\lambda}}{\varkappa^{(2+\delta)/\delta}} = T$,

$$\left|f_\lambda(t) - e^{-t^2/2}\right| \leqslant \frac{2^{1-\delta}}{(1+\delta)(2+\delta)}\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}|t|^{2+\delta}e^{-t^2/4}, \tag{7}$$

$$\left|f_\lambda'(t) - \left(e^{-t^2/2}\right)'\right| \leqslant \frac{2^{1-\delta}}{1+\delta}\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}|t|^{1+\delta}e^{-t^2/2}$$

$$+ \left\{\frac{2^{1-\delta}}{(1+\delta)(2+\delta)}\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}|t|^{3+\delta} + \frac{2^{2(1-\delta)}}{(1+\delta)^2(2+\delta)}\left(\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}\right)^2|t|^{3+2\delta}\right\}e^{-t^2/4}, \tag{8}$$

where $C_0(\delta) = \left(\frac{(1+\delta)(2+\delta)}{2^{3-\delta}}\right)^{1/\delta}$.

Proof. It is easy to see that $\mathbb{E}S_{N_\lambda} = \lambda\mu$, $\mathbb{D}S_{N_\lambda} = \lambda(\mu^2 + \sigma^2) = \lambda\alpha_2$, and

$$f_\lambda(t) = \exp\left\{\lambda\left(\mathbb{E}e^{i\frac{t}{\sqrt{\lambda\alpha_2}}X} - 1 - i\frac{t}{\sqrt{\lambda\alpha_2}}\mu\right)\right\}. \tag{9}$$

To make sure on the correctness of (6), it suffices to take the derivatives with respect to t on both sides of expression (9).

Denote by θ_1 and θ_2 complex functions such that $|\theta_i| \leqslant 1$.

Estimation of $|f_\lambda(t) - e^{-t^2/2}|$. Since for all $x \in \mathbb{R}$, $0 < \delta \leqslant 1$, and $i = \sqrt{-1}$,

$$\left|e^{ix} - 1 - ix\right| \leqslant \frac{2^{1-\delta}}{1+\delta}|x|^{1+\delta},$$

we derive from (6) that, for all $t \in \mathbb{R}$,

$$f'_\lambda(t) = \left(-t + \theta_1\frac{2^{1-\delta}}{1+\delta}\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}|t|^{1+\delta}\right)f_\lambda(t). \tag{10}$$

After solving the linear differential equation (10), we get that the solution of Eq. (10) can be written in the form

$$f_\lambda(t) = \exp\left\{-\frac{t^2}{2} + \theta_2\frac{2^{1-\delta}}{(1+\delta)(2+\delta)}\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}|t|^{2+\delta}\right\}, \quad t \in \mathbb{R}.$$

To estimate the difference $|f_\lambda(t) - e^{-t^2/2}|$, we use the well-known fact that $|e^z - 1| \leqslant |z|e^{|z|}$ for all complex numbers z. We obtain that for all $|t| \leqslant C_0(\delta)\frac{\sqrt{\lambda}}{\varkappa^{(2+\delta)/\delta}}$

$$\left|f_\lambda(t) - e^{-t^2/2}\right| \leqslant \frac{2^{1-\delta}}{(1+\delta)(2+\delta)}\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}|t|^{2+\delta}e^{-t^2/4},$$

i.e., (7) is proved.

Substituting (7) into (10), we get (8).

Lemma 1 is proved.

3. Proof of Theorem 1

Estimation of $\Delta_{\lambda,\infty}$. To estimate the uniform metric $\Delta_{\lambda,\infty}$, we use the smoothing inequality of Esséen [12, p. 297] with $T = C_0(\delta)\frac{\sqrt{\lambda}}{\varkappa^{(2+\delta)/\delta}} > 0$, inequality (7), and obtain

$$\Delta_{\lambda,\infty} \leqslant \frac{2}{\pi}\int_0^T \left|\frac{f_\lambda(t) - e^{-t^2/2}}{t}\right|dt + \frac{24}{\pi\sqrt{2\pi}}\frac{1}{T} \leqslant C(\delta)\left(\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}} + \left(\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}\right)^{1/\delta}\right).$$

Let $\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}} \leqslant 1$. Then $\left(\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}\right)^{1/\delta} \leqslant \frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}$ and $\Delta_{\lambda,\infty} \leqslant C(\delta)\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}$. In the case $\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}} > 1$, the estimate of $\Delta_{\lambda,\infty}$ is trivial since $\Delta_{\lambda,\infty} \leqslant 0.5416$ [2, p. 103].

Estimation of $\Delta_{\lambda,1}$. To estimate the L_1 norm $\Delta_{\lambda,1}$, we use the estimates of Lemma 1 and the following inequality with $T \geqslant 1$ ([8, p. 25] and [13, p. 395]):

$$\int_{-\infty}^{\infty} \left| F_{Z_{N_\lambda}}(x) - \Phi(x) \right| dx$$

$$\leqslant 3\left(\int_0^T \left| \frac{f_\lambda(t) - e^{-t^2/2}}{t} \right|^2 dt \right)^{1/2} + \sqrt{2}\left(\int_0^T \left| \frac{d}{dt}\left(\frac{f_\lambda(t) - e^{-t^2/2}}{t} \right) \right|^2 dt \right)^{1/2} + \frac{8\pi}{T}$$

$$\leqslant 3I_1 + 2(I_2 + I_3) + \frac{8\pi}{T}, \tag{11}$$

where

$$I_1^2 = \int_0^T \left| \frac{f_\lambda(t) - e^{-t^2/2}}{t} \right|^2 dt, \quad I_2^2 = \int_0^T \left| \frac{f_\lambda'(t) - (e^{-t^2/2})'}{t} \right|^2 dt,$$

$$I_3^2 = \int_0^T \left| \frac{f_\lambda(t) - e^{-t^2/2}}{t^2} \right|^2 dt.$$

Using inequalities (7) and (8), we estimate the quantities I_1, I_2, and I_3 from (11) with $T = C_0(\delta)\frac{\sqrt{\lambda}}{\varkappa^{(2+\delta)/\delta}} > 1$, and obtain

$$I_1 \leqslant C(\delta)\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}, \quad I_2 \leqslant C(\delta)\left(\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}} + \left(\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}} \right)^2 \right), \quad I_3 \leqslant C(\delta)\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}.$$

To estimate I_2, we use the elementary inequality $(a + b + c)^2 \leqslant 3(a^2 + b^2 + c^2)$. Substituting these estimates into (11), we have that, for $T = C_0(\delta)\frac{\sqrt{\lambda}}{\varkappa^{(2+\delta)/\delta}} > 1$ and $\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}} \leqslant 1$,

$$\Delta_{\lambda,1} \leqslant 3I_1 + 2(I_2 + I_3) + \frac{8\pi}{T} \leqslant C(\delta)\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}}.$$

In the case $\frac{\varkappa^{2+\delta}}{\lambda^{\delta/2}} > 1$ (in the case $C_0(\delta)\frac{\sqrt{\lambda}}{\varkappa^{(2+\delta)/\delta}} \leqslant 1$ as well), the estimate of $\Delta_{\lambda,1}$ is trivial since $\Delta_{\lambda,p} \leqslant \sqrt{2}$ for all $1 \leqslant p \leqslant \infty$ (for $\Delta_{\lambda,1} \leqslant \sqrt{2}$, see [6, p. 528]).

The estimate (5) now follows from the estimates of $\Delta_{\lambda,\infty}$ and $\Delta_{\lambda,1}$, since

$$\Delta_{\lambda,p} \leqslant \Delta_{\lambda,\infty}^{(p-1)/p} \Delta_{\lambda,1}^{1/p}$$

for all $1 \leqslant p < \infty$.

Theorem 1 is proved.

References

1. V.E. Bening, V.Yu. Korolev, S.Ya. Shorgin. On approximations to generalized Poisson distributions. *J. Math. Sci.*, 83(3):360–373, 1997.

2. R.N. Bhattacharya, R. Ranga Rao. *Normal Approximation and Asymptotic Expansions*. Krieger Publishing Company, Malabar, Florida, 1986.

3. R. Von Chossy, G. Rappl. Some approximation methods for the distribution of random sums. *Insurance*: *Math. Econom.*, 2:251–270, 1983.

4. G. Englund. A remainder term estimate in a random-sum central limit theorem. *Teor. Veroyatn. Primen.*, 28(1):143–149, 1983.

5. R.V. Erickson. On an L_p version of the Berry–Esséen theorem for independent and m-dependent random variables. *Ann. Probab.*, 1(3):497–503, 1973.

6. R.V. Erickson. L_1 bounds for asymptotic normality of m-dependent sums using Stein's technique. *Ann. Probab.*, 2:522–529, 1974.

7. B.V. Gnedenko, V.Yu. Korolev. *Random Summation. Limit Theorems and Applications*. CRC Press, Boca Raton, FL, 1996.

8. I.A. Ibragimov, Yu.V. Linnik. *Independent and Stationary Connected Variables*. Nauka, Moskva, 1965 (in Russian).

9. V.Yu. Korolev. O tochnosti normal'noj approksimacii dlja raspredelenij summ sluchajnogo chisla nezavisimyh sluchajnyh velichin. *Teor. Veroyatn. Primen.*, 33(3):577–581, 1988.

10. V.Yu. Korolev, S.Ya. Shorgin. On the absolute constant in the remainder term estimate in the central limit theorem for Poisson random sums. In: *Probabilistic Methods in Discrete Mathematics*, *Proc. 4th Internat. Petrozavodsk Conference*. VSP, Utrecht, 305–308, 1997.

11. V.M. Kruglov, V.Yu. Korolev. *Predelnye teoremy dlja sluchajnyh summ*. Izd. Mosk. univ., Moskva, 1990.

12. M. Loève. *Probability Theory*, I. 4th edition. Springer-Verlag, New York, Berlin, Heidelberg, 1977.

13. E. Lukacs. *Characteristic Functions*. Nauka, Moskva, 1979 (in Russian).

14. R. Michel. On Berry–Esseen results for the compound Poisson distribution. *Insurance*: *Math. Econom.*, 13(1):35–37, 1986.

15. G.V. Rotar. On one problem of storages control. *Econ. Math. Methods*, 12(4):733–739, 1976.

16. H. Robbins. The asymptotic distribution of the sum of a random number of random variables. *Bull. Amer. Math. Soc.*, 54:1151–1161, 1948.

17. I.S. Shiganov. O tochnosti verhnej ocenki konstanty v ostatochnom chlene central'noj priedel'noj teoremy. In: *Problemy ustojchivosti stohasticheskih modelej*, *Trudy seminara VNIISI*, 109–115, 1982.

18. Ch. Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In: *Proc. Math. Statist. and Probab.*, Vol. 2, Univ. Calif. Press, Berkeley, CA, 583–602, 1972.

19. J. Sunklodas. On the rate of convergence of L_p norms in the central limit theorem for independent random variables. *Lith. Math. J.*, 42(3), 296–307, 2002.

20. J. Sunklodas. On the rate of convergence of L_p norms in the CLT for Poisson and gamma random variables. *Lith. Math. J.*, 49(2):216–221, 2009.

21. A.N. Tikhomirov. On the rate of convergence in the central limit theorem for weakly dependent variables. *Theory Probab. Appl.*, 25:790–809, 1980.

REZIUMĖ

J. Sunklodas. Apie normos L_p konvergavimo greitį CRT Puasono atsitiktinei sumai

Gautas L_p normos $\Delta_{\lambda,p}$ viršutinis įvertis eilės $\lambda^{-\delta/2}$ su visais $1 \leqslant p \leqslant \infty$ centrinėje ribinėje teoremoje standartizuotai sumai $(S_{N_\lambda} - \mathbb{E}S_{N_\lambda})/\sqrt{\mathbb{D}S_{N_\lambda}}$, kur $S_{N_\lambda} = X_1 + \cdots + X_{N_\lambda}$ yra suma nepriklausomų vienodai pasiskirsčiusių atsitiktinių dydžių $X, X_1, X_2, \ldots$ su $\beta_{2+\delta} = \mathbb{E}|X|^{2+\delta} < \infty$ nekuriam $0 < \delta \leqslant 1$, N_λ yra Puasono atsitiktinis dydis su parametru $\lambda > 0$ ir N_λ nepriklauso nuo atsitiktinių dydžių $X_1, X_2, \ldots$.

Raktiniai žodžiai: centrinė ribinė teorema, L_p norma, Puasono atsitiktinė suma.