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Exact bounds for tail probabilities of martingales with bounded differences

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Abstract. We consider random walks, say $W_n = \{0, M_1, \dots, M_n\}$ of length n starting at 0 and based on a martingale sequence $M_k = X_1 + \cdots + X_k$ with differences X_m. Assuming $|X_k| \leqslant 1$ we solve the isoperimetric problem

$$B_n(x) = \sup \mathbb{P}\bigl\{W_n \text{ visits an interval } [x, \infty)\bigr\}, \tag{1}$$

where sup is taken over all possible W_n. We describe random walks which maximize the probability in (1). We also extend the results to super-martingales. For martingales our results can be interpreted as a maximal inequalities

$$\mathbb{P}\Bigl\{\max_{1 \leqslant k \leqslant n} M_k \geqslant x\Bigr\} \leqslant B_n(x).$$

The maximal inequality is optimal since the equality is achieved by martingales related to the maximizing random walks. To prove the result we introduce a general principle – maximal inequalities for (natural classes of) martingales are equivalent to (seemingly weaker) inequalities for tail probabilities, in our case

$$B_n(x) = \sup \mathbb{P}\{M_n \geqslant x\}.$$

Our methods are similar in spirit to a method used in [1], where a solution of an isoperimetric problem (1), for integer x is provided and to the method used in [4], where the isoperimetric problem of type (1) for conditionally symmetric bounded martingales was solved for all $x \in \mathbb{R}$.

Keywords: random walks, maximal inequalities, probability to visit an interval, large deviations, martingale, super-martingale, bounds for tail probabilities.

1. Introduction

We start with a simple case of martingales with bounded differences. Extensions to super-martingales are provided at Section 2.

Firstly we consider random walks, say $W_n = \{0, M_1, \dots, M_n\}$ of length n starting at 0 and based on a martingale sequence $M_k = X_1 \dots X_k$ with differences $X_m = M_m - M_{m-1}$. Let $\mathcal{M}_1$ be the class of martingales with bounded differences such that $|X_m| \leqslant 1$, and let $\mathcal{M}_{1,\mathrm{sym}}$ stands for the subclass of $\mathcal{M}_1$ of martingales with the conditionally symmetric differences such that $\mathbb{P}\{X_m \in A \mid X_1, \dots, X_{m-1}\} =$

$\mathbb{P}\{X_m \in -A\,\{|\,X_1,\ldots,X_{m-1}\}$ for all measurable non-random sets A and all allowable m. If a random walk W_n is based on a martingale sequence of the class $\mathcal{M}_1$ (respectively, of the class $\mathcal{M}_{1,\mathrm{sym}}$) then we write symbolically $W_n \in \mathcal{M}_1$ (respectively, $W_n \in \mathcal{M}_{1,sym}$).

We provide a solution of the isoperimetric (or variational) problem

$$B_n(x) = \sup_{W_n \in \mathcal{M}_1} \mathbb{P}\big\{W_n \text{ visits an interval } [x,\infty)\big\}, \quad x \in \mathbb{R}. \tag{2}$$

In particular, we describe random walks which maximize the probability in (2). It turns out that for integer $x \in \mathbb{Z}$ a random walk maximizing the probability in (2) is a simple symmetric random walk (that is, a symmetric random walk with independent steps of length 1). For non-integer x, the maximizing random walk has as a component some auxiliary random walks with independent steps of smaller sizes. The average total number of the components are both bounded by 2. For martingales our result can be interpreted as a maximal inequality

$$P\Big\{\max_{1\leqslant k\leqslant n} M_k \geqslant x\Big\} \leqslant B_n(x).$$

The maximal inequality is optimal since the equality is achieved by martingales related to the maximizing random walks, that is,

$$\sup_{M_1,\ldots,M_n \in \mathcal{M}_1} \mathbb{P}\Big\{\max_{1\leqslant k\leqslant n} M_k \geqslant x\Big\} = B_n(x).$$

To prove the result we introduce a general principle – maximal inequalities for (natural classes of) martingales are equivalent to (seemingly weaker) inequalities for tail probabilities, which reads as

$$\sup_{M_1,\ldots,M_n \in \mathcal{M}_{1,}} \mathbb{P}\Big\{\max_{1\leqslant k\leqslant n} M_k \geqslant x\Big\} = \sup_{M_n \in \mathcal{M}_{1,}} \mathbb{P}\{M_n \geqslant x\} \tag{3}$$

in our particular case. Our methods are similar in spirit to a method used in [1], where a solution of an isoperimetric problem 2 was provided for integer $x \in \mathbb{Z}$ and to the methods used in the article [4], where the isoperimetric problem

$$D_n(x) = \sup_{W_n \in \mathcal{M}_{1,\mathrm{sym}}} \mathbb{P}\big\{W_n \text{ visits an interval } [x,\infty)\big\}$$

was solved for all $x \in R$.

A great number of papers is devoted to construction of *upper* bounds for the functions of type $D_n(x)$ or $B_n(x)$. For most advanced results, as well as for a review of the related literature see [2].

Let us now provide a description of the random walks maximizing the probability in (1). For given $0 < x \leqslant n$ and n, we introduce a random walk, say $W_{n,x}$ as follows. Let us describe the rules regulating the random walk. We start from 0, hence our start position is $M_0 = 0$. Let our position after k steps be M_k, and let $\varrho_k = x - M_k$ be the remaining distance between our position and our "target" x. Then the next step depends on the following four possible situations:

i) $\varrho_k \in \mathbb{Z}$;
ii) $\varrho_k \notin \mathbb{Z}$, $n - k - [\varrho_k] \in 2\mathbb{Z} + 1$;
iii) $\varrho_k \notin \mathbb{Z}$, $n - k - [\varrho_k] \in 2\mathbb{Z}$, $\varrho_k > 1$;
iv) $\varrho_k \notin \mathbb{Z}$, $n - k - [\varrho_k] \in 2\mathbb{Z}$, $0 < \varrho_k < 1$.

In all four situations we make a step to right or left, so we have to define only the sizes of the steps, say s_r and s_l.

i) $s_r = s_l = 1$, that is, X_{k+1} is an independent copy of the Rademacher r.v. ε.
ii) $s_l = 1$, $s_r = \{\varrho_k\}$, that is, it jumps to the right on the integer grid, or to the left by the maximum step.
iii) $s_l = 1 - \{\varrho_k\}$, $s_r = 1$, that is, it jumps to the left on the integer grid, or to the right by the maximum step.
iv) $s_l = 1 - \{\varrho_k\}$, $s_r = \{\varrho_k\}$, that is, it jumps to the left and right on the integer grid.

Here $[x]$ and $\{x\}$ denotes the integer and fractional parts of a number $x \in \mathbb{R}$.

From this we can see, that the random walk makes in average only 2 smaller than 1 steps.

2. An extension to super-martingales

We call a random variable U super-symmetric (respectively sub-symmetric) if $\mathbb{P}\{U \in A\} \leqslant P\{U \in -A\}$ (respectively $\mathbb{P}\{U \in A\} \geqslant \mathbb{P}\{U \in -A\}$) for all measurable subsets A of $[0, \infty]$. It is clear that a random variable is symmetric if and only if it is sub-symmetric and super-symmetric.

Let $\mathcal{SM}_1$ be the class of super-martingales with bounded differences such that $|X_m| \leqslant 1$, and let $\mathcal{SM}_{1,\text{super}}$ stands for the subclass of $\mathcal{SM}_1$ of super-martingales with the conditionally super-symmetric differences such that

$$\mathbb{P}\bigl\{X_m \in A \mid X_1, \dots, X_{m-1}\bigr\} \leqslant \mathbb{P}\bigl\{X_m \in -A \mid X_1, \dots, X_{m-1}\bigr\}, \tag{4}$$

for all measurable non-random subsets A of $[0, 1]$ and all allowable m. Note that $\mathcal{M}_{1,\text{sym}} \subset \mathcal{SM}_{1,\text{super}}$.

We show that

$$D_n(x) = \sup_{W_n \in \mathcal{SM}_{1,\text{super}}} \mathbb{P}\bigl\{W_n \text{ visits an interval } [x, \infty)\bigr\}, \quad x \in \mathbb{R} \tag{5}$$

and

$$B_n(x) = \sup_{W_n \in \mathcal{SM}_1} \mathbb{P}\bigl\{W_n \text{ visits an interval } [x, \infty)\bigr\}, \quad x \in \mathbb{R}. \tag{6}$$

For super-martingales and super-symmetric super-martingales these results as well can be interpreted as the maximal inequalities

$$\mathbb{P}\Bigl\{\max_{1 \leqslant k \leqslant n} M_{k,sym} \geqslant x\Bigr\} \leqslant D_n(x),$$

$$\mathbb{P}\Bigl\{\max_{1 \leqslant k \leqslant n} M_k \geqslant x\Bigr\} \leqslant B_n(x),$$

where $M_{k,sym} \in \mathcal{SM}_{1,\text{super}}$ and $M_k \in \mathcal{SM}_1$. Furthermore, the sup over the class of super-martingales is achieved on a martingale class.

References

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REZIUMĖ

D. Dzindzalieta. Tikslūs rėžiai martingalams su aprėžtais skirtumais

Šiame darbe sprendžiamas variacinis uždavinys

$$B_n(x) \overset{\text{def}}{=} \sup_{M_n \in \mathcal{M}_n} \mathbb{P}\{M_n \geqslant x\},$$

visiems $x \in \mathbb{R}$, kur $\mathcal{M}_n$ yra klasė aprėžtų martingalų.

Martingalo M_n skirtumai X_k tenkina sąlygą $\mathbb{E}(X_k \mid \mathfrak{F}_{k-1}) = 0$ didėjančios σ-algebrų iš mačiosios erdvės $(\Omega, \mathfrak{F})$, šeimos $\emptyset = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}n \subset \mathfrak{F}$ atžvilgiu. Tariama, kad skirtumai yra aprėžti (t.y. $|X_k| \leqslant 1$). Mūsų metodai iš esmės panašūs į metodą, kurį naudojo Bentkus 2001 rėžiui (2) gauti, kai $x \in \mathbb{Z}$ ir metodą, kurį naudojo D 2006 panašaus uždavinio rezultatui su sąlyginai simetriniais skirtumais gauti.

Rezultatas gali būti pritaikomas apibūdinti atsitiktiniams klajojimams, kurie maksimizuoja tikimybę patekti i intervalą, kai kuriems mato koncentracijos dominavimo modeliams, atsitiktinių grafų teorijoje ir pan.

Interpretuoti $W_n = \{0, X_1, X_1 + X_2, \dots, X_1 + \cdots + X_n\} = \{0, M_1, \dots, M_n\}$ galima kaip n žingsnių atsitiktinį klajojimą prasidedanti nulyje. Tegul $P_x(W_n)$ yra tikimybė aplankyti intervalą $[x, \infty]$ per pirmus n žingsnių, t.y.

$$P_x(W_n) = \mathbb{P}\{\max_{0 \leqslant k \leqslant n} M_k \geqslant x\}.$$

Sveikiesiems x uždavinį (1) galime performuluoti kaip izoperimetrinį uždavinį

$$B_n(x) = P_x(RW_n) = \sup_{W_n} \mathbb{P}_x(W_n),$$

kur $RW_n = \{0, \varepsilon_1, \varepsilon_1 + \varepsilon_2, \dots, \varepsilon_1 + \cdots + \varepsilon_n\}$ žymimas simetrinis paprastasis atsitiktinis klajojimas prasidedantis taške 0 (čia $\varepsilon, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ yra nepriklausomi vienodai pasiskirstę Rademacherio atsitiktiniai dydžiai, t.y. $\mathbb{P}\{\varepsilon = 1\} = \mathbb{P}\{\varepsilon = -1\} = \frac{1}{2}$). Kitais žodžiais, tarp visų atsitiktinių klajojimų su aprėžtais žingsnių ilgiais ir tarp visų aprėžtų atsitiktinių klajojimų, simetrinis paprastasis atsitiktinis klajojimas maksimizuoja tikimybę patekti į intervalą $[x, \infty]$, kur $x \in \mathbb{Z}$.

Raktiniai žodžiai: atsitiktiniai klajojimai, tikimybė patekti į intervalą, martingalas, supermartingalas, rėžiai uodegų tikimybėms.