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Orthogonal decomposition of finite population L-statistics

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Abstract. In this paper we study orthogonal decomposition of finite population L-statistics. We propose quite simple form of first two terms of such decomposition.

Keywords: finite population, sampling without replacement, L-statistic, Hoeffding decomposition.

1. Introduction

Consider the population $\mathcal{X} = \{x_1, \dots, x_N\}$ of size N and assume that $x_1 < \dots < x_N$. Let $X_1, \dots, X_n$ is simple random sample of size $n < N$ drawn without replacement from $\mathcal{X}$ and let $X_{(1)} < \dots < X_{(n)}$ denote the order statistics of $X_1, \dots, X_n$. Then for arbitrary real numbers $c_1, \dots, c_n$ define L-statistic $L = c_1 X_{(1)} + \dots + c_n X_{(n)}$.

This statistic can be decomposed into the sum

$$L = \mathbf{E}L + U_1 + \dots + U_n, \tag{1}$$

$$U_m = \sum_{1 \leqslant i_1 < \dots < i_m \leqslant n} g_m(X_{i_1}, \dots, X_{i_m}), \quad m = 1, \dots, n.$$

Decomposition (1) is called orthogonal (called also Hoeffding) decomposition and U_m are called U-statistics. The symmetric kernels $g_m, m = 1, \dots, n$ are linear combinations of conditional expectations

$$h_j(x_{k_1}, \dots, x_{k_j}) = \mathbf{E}(L - \mathbf{E}L | X_1 = x_{k_1}, \dots, X_j = x_{k_j}), \quad 1 \leqslant j \leqslant m. \tag{2}$$

The decompositon (1) and its applications for finite population symmetric statistics were studied in [1]. In that paper we can find coefficients of linear combinations of (2).

The main interest of present work is conditional expectations (2) for $j = 1, \dots, n$ and functions

$$g_1(x) = \frac{N-1}{N-n} h_1(x), \tag{3}$$

$$g_2(x, y) = \frac{N-2}{N-n} \frac{N-3}{N-n-1} \left(h_2(x, y) - \frac{N-1}{N-2} (h_1(x) + h_1(y)) \right), \tag{4}$$

which can be useful for various applications.

In the case where random variables $X_1, \ldots, X_n$ are independent and identically distributed the orthogonal decomposition of L-statistics was studied in [2]. We shall adopt some ideas from that paper to get a similar form of orthogonal decomposition. Note that for samples without replacement from finite population variables $X_1, \ldots, X_n$ are identically distributed, but they are not independent.

In Section 2 we state our results, which are proved in Section 3.

2. Results

For the given n let fix $0 \leqslant m \leqslant n$ and define a set of conditions $A_m = \{X_1 = x_{k_1}, \ldots, X_m = x_{k_m}\}$, where $1 \leqslant k_1 < \ldots < k_m \leqslant N$. Let $k_0 = 0$, $k_{m+1} = N + 1$ and define $X_{(0)} = x_0$, $X_{(n+1)} = x_{N+1}$ where $x_0 = x_1$, $x_{N+1} = x_N$. Consider statistics $X_{(r+1)} - X_{(r)}$, $r = 0, \ldots, n$.

LEMMA 1. *For any $m = 0, \ldots, n$ and $r = 0, \ldots, n$ we have*

$$\mathbf{E}(X_{(r+1)} - X_{(r)}|A_m) = \sum_{s=1}^{m+1} \sum_{i=k_{s-1}}^{k_s-1} \Delta_{m,s,i}(r)(x_{i+1} - x_i), \tag{5}$$

where we denote

$$\Delta_{m,s,i}(r) = \binom{N-m}{n-m}^{-1} \binom{i-s+1}{r-s+1} \binom{N-i-m+s-1}{n-r-m+s-1}.$$

We shall use differences $X_{(r+1)} - X_{(r)}$, $r = 0, \ldots, n$ to get convenient expression of (2).

PROPOSITION 1. *For chosen $m = 1, \ldots, n$ we have*

$$\mathbf{E}(L - \mathbf{E}L|A_m) = \sum_{j=1}^{n} c_j \sum_{r=0}^{j-1} \big\{\mathbf{E}\big(X_{(r+1)} - X_{(r)}|A_m\big) - \mathbf{E}\big(X_{(r+1)} - X_{(r)}\big)\big\}. \tag{6}$$

Next we shall propose simple form of kernels (3) and (4).

THEOREM 1. (i) *For $1 \leqslant k \leqslant N$*

$$g_1(x_k) = -\sum_{j=1}^{n} c_j \sum_{i=1}^{N-1} \varphi_k(i) \frac{\binom{i-1}{j-1}\binom{N-i-1}{n-j}}{\binom{N-2}{n-1}} (x_{i+1} - x_i), \tag{7}$$

where

$$\varphi_k(i) = \begin{cases} -\frac{i}{N}, & \text{if } 1 \leqslant i < k, \\ 1 - \frac{i}{N}, & \text{if } k \leqslant i < N. \end{cases}$$

(ii) *For* $1 \leqslant k < l \leqslant N$

$$g_2(x_k, x_l) = -\sum_{j=2}^{n}(c_j - c_{j-1}) \sum_{i=1}^{N-1} \phi_{k,l}(i) \frac{\binom{i-2}{j-2}\binom{N-i-2}{n-j}}{\binom{N-4}{n-2}}(x_{i+1} - x_i), \tag{8}$$

where

$$\phi_{k,l}(i) = \begin{cases} \frac{i(i-1)}{(N-1)(N-2)}, & \text{if } 1 \leqslant i < k, \\ -\frac{(i-1)(N-i-1)}{(N-1)(N-2)}, & \text{if } k \leqslant i < l, \\ \frac{(N-i)(N-i-1)}{(N-1)(N-2)}, & \text{if } l \leqslant i < N. \end{cases}$$

Remark 1. While we do not talk about applications, which are beyond the scope of this paper, the Theorem 1 is just a theoretical result, i.e., the kernels (7) and (8) are just the formal functions.

3. Proofs

Proof of Lemma 1. For any $m = 0, \ldots, n$ and $r = 0, \ldots, n+1$ straightforward combinatorial calculations give

$$\mathbf{E}(X_{(r)}|A_m) = \binom{N-m}{n-m}^{-1} \Bigg[\sum_{s=1}^{m+1} \sum_{i=k_{s-1}+1}^{k_s - 1} \binom{i-s}{r-s}\binom{N-i-m+s-1}{n-r-m+s-1} x_i + \sum_{s=0}^{m+1} \binom{k_s - s}{r-s}\binom{N-k_s-m+s}{n-r-m+s} x_{k_s} \Bigg].$$

The key idea is for $r = 0, \ldots, n$ write

$$\mathbf{E}(X_{(r+1)}|A_m) = \binom{N-m}{n-m}^{-1} \Bigg[\sum_{s=1}^{m+1} \sum_{i=k_{s-1}+1}^{k_s - 1} \binom{i-s}{r-s+1} \delta'_{m,s,i}(r) x_i + \sum_{s=0}^{m+1} \binom{k_s - s}{r-s+1}\binom{N-k_s-m+s}{n-r-m+s-1} x_{k_s} \Bigg],$$

where

$$\delta'_{m,s,i}(r) = \binom{N-i-m+s}{n-r-m+s-1} - \binom{N-i-m+s-1}{n-r-m+s-1}$$

and

$$\mathbf{E}(X_{(r)}|A_m) = \binom{N-m}{n-m}^{-1} \Bigg[\sum_{s=1}^{m+1} \sum_{i=k_{s-1}+1}^{k_s - 1} \delta''_{m,s,i}(r) \binom{N-i-m+s-1}{n-r-m+s-1} x_i$$

$$+\sum_{s=0}^{m+1}\binom{k_s-s}{r-s}\binom{N-k_s-m+s}{n-r-m+s}x_{k_s}\Bigg],$$

where

$$\delta''_{m,s,i}(r)=\binom{i-s+1}{r-s+1}-\binom{i-s}{r-s+1}.$$

Then it is easy to see, that for $r=0,\dots,n$, $\mathbf{E}(X_{(r+1)}-X_{(r)}|A_m)$ is the same as in lemma's statement.

Proof of Proposition 1. Applying summation by parts we can write

$$L=\sum_{r=1}^{n-1}\alpha_r\big(X_{(r+1)}-X_{(r)}\big)+\bar{c}\sum_{j=1}^{n}X_j,$$

where $\alpha_r=-\sum_{j=1}^{r}(c_j-\bar{c})$ for $r=1,\dots,n-1$ and $\bar{c}=\frac{1}{n}\sum_{j=1}^{n}c_j$.

Then for $m=1,\dots,n$ we have

$$\begin{aligned}\mathbf{E}(L-\mathbf{E}L|A_m)=&-\sum_{j=1}^{n}c_j\sum_{r=j}^{n}\big\{\mathbf{E}\big(X_{(r+1)}-X_{(r)}|A_m\big)-\mathbf{E}\big(X_{(r+1)}-X_{(r)}\big)\big\}\\&+\bar{c}\Bigg[\sum_{r=0}^{n}r\big\{\mathbf{E}\big(X_{(r+1)}-X_{(r)}|A_m\big)-\mathbf{E}\big(X_{(r+1)}-X_{(r)}\big)\big\}\\&+\frac{N-n}{N(N-m)}\sum_{s=1}^{m}\Bigg(\sum_{i=0}^{k_s-1}i(x_{i+1}-x_i)-\sum_{i=k_s}^{N}(N-i)(x_{i+1}-x_i)\Bigg)\Bigg].\end{aligned}$$

Note that the term in brackets vanishes, because using lemma 1 and changing order of summation, for fixed $m=0,\dots,n$, $s=1,\dots,m+1$, $i=k_{s-1},\dots k_s-1$

$$\begin{aligned}\sum_{r=0}^{n}r\Delta_{m,s,i}(r)&=\sum_{r=s-1}^{n-m+s-1}(r-s+1)\Delta_{m,s,i}(r)+(s-1)\sum_{r=s-1}^{n-m+s-1}\Delta_{m,s,i}(r)\\&=\frac{n-m}{N-m}(i-s+1)+s-1,\end{aligned}$$

where

$$\Delta_{m,s,i}(r)=\binom{N-m}{n-m}^{-1}\binom{i-s+1}{r-s+1}\binom{N-m-(i-s+1)}{n-m-(r-s+1)},$$

and the remaining verifying is quite simple.

Applying of Vandermonde's identity completes the proof.

Proof of Theorem 1. (i) For chosen $1 \leqslant k \leqslant N$ using proposition 1 for $m = 1$ and lemma 1 for $m = 0; 1$ from (3) we have

$$g_1(x_k) = \binom{N-2}{n-1}^{-1} \sum_{j=1}^{n} c_j \sum_{r=0}^{j-1} \left\{ \sum_{i=1}^{k-1} \frac{i}{N} \theta_{21}(i,r)(x_{i+1} - x_i) - \sum_{i=k}^{N-1} \left(1 - \frac{i}{N}\right) \theta_{22}(i,r)(x_{i+1} - x_i) \right\},$$

where

$$\theta_{21}(i,r) = \frac{N}{i}\binom{i}{r}\left\{\binom{N-i-1}{n-r-1} - \frac{n}{N}\binom{N-i}{n-r}\right\},$$

$$\theta_{22}(i,r) = -\frac{N}{N-i}\binom{N-i}{n-r}\left\{\binom{i-1}{r-1} - \frac{n}{N}\binom{i}{r}\right\}.$$

It is easy to verify that $\theta_{21}(i,r) = \theta_{22}(i,r)$. Next using principle of mathematical induction it is easy to show that for every $j = 1, \ldots, n$

$$\sum_{r=0}^{j-1} \theta_{22}(i,r) = \binom{i-1}{j-1}\binom{N-i-1}{n-j},$$

and the proof of the part (i) follows.

(ii) For chosen $1 \leqslant k < l \leqslant N$ using Proposition 1 for $m = 1; 2$ and Lemma 1 for $m = 0; 1; 2$ from (4) we have

$$g_2(x_k, x_l) = \binom{N-4}{n-2}^{-1} \sum_{j=1}^{n} c_j \sum_{r=0}^{j-1} \left\{ \sum_{i=1}^{k-1} \frac{i(i-1)}{(N-1)(N-2)} \theta_{31}(i,r)(x_{i+1} - x_i) - \sum_{i=k}^{l-1} \frac{(i-1)(N-i-1)}{(N-1)(N-2)} \theta_{32}(i,r)(x_{i+1} - x_i) + \sum_{i=l}^{N-1} \frac{(N-i)(N-i-1)}{(N-1)(N-2)} \theta_{33}(i,r)(x_{i+1} - x_i) \right\},$$

where

$$\theta_{31}(i,r) = \frac{(N-1)(N-2)}{i(i-1)}\binom{i}{r}\left\{\binom{N-i-2}{n-r-2} - 2\frac{n-1}{N-2}\binom{N-i-1}{n-r-1} + \frac{n(n-1)}{(N-1)(N-2)}\binom{N-i}{n-r}\right\},$$

$$\theta_{32}(i,r) = -\frac{(N-1)(N-2)}{(i-1)(N-i-1)}\Bigg[\binom{i-1}{r-1}\Bigg\{\binom{N-i-1}{n-r-1} - \frac{n-1}{N-2}\binom{N-i}{n-r}\Bigg\}$$
$$- \frac{n-1}{N-2}\binom{i}{r}\Bigg\{\binom{N-i-1}{n-r-1} - \frac{n}{N-1}\binom{N-i}{n-r}\Bigg\}\Bigg],$$
$$\theta_{33}(i,r) = \frac{(N-1)(N-2)}{(N-i)(N-i-1)}\binom{N-i}{n-r}\Bigg\{\binom{i-2}{r-2} - 2\frac{n-1}{N-2}\binom{i-1}{r-1}$$
$$+ \frac{n(n-1)}{(N-1)(N-2)}\binom{i}{r}\Bigg\}.$$

Similarly $\theta_{31}(i,r) = \theta_{32}(i,r) = \theta_{33}(i,r)$. Next using principle of mathematical induction we can show that for every $j = 1, \ldots, n$

$$\sum_{r=0}^{j-1}\theta_{33}(i,r) = \binom{i-2}{j-1}\binom{N-i-2}{n-j-1} - \binom{i-2}{j-2}\binom{N-i-2}{n-j}.$$

Then summation by parts completes the proof of the part (ii).

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References

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REZIUMĖ

A. Čiginas. Baigtinių populiacijų L-statistikų ortogonalusis skleidinys

Straipsnyje nagrinėjamas baigtinių populiacijų L-statistikų ortogonalusis skleidinys. Pasiūlomos patogios pirmųjų dviejų skleidinio narių išraiškos.

Raktiniai žodžiai: baigtinė populiacija, ėmimas be grąžinimo, L-statistika, Hoeffding'o skleidinys.