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Estimating the Hurst index of the solution of a stochastic integral equation

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Abstract. Let $X(t)$ be a solution of a stochastic integral equation driven by fractional Brownian motion B^H and let $V_n^2(X,2)=\sum_{k=1}^{n-1}(\Delta_k^2 X)^2$ be the second order quadratic variation, where $\Delta_k^2 X = X\big(\frac{k+1}{n}\big)-2X\big(\frac{k}{n}\big)+X\big(\frac{k-1}{n}\big)$. Conditions under which $n^{2H-1}V_n^2(X,2)$ converges almost surely as $n\to\infty$ was obtained. This fact is used to get a strongly consistent estimator of the Hurst index H, $1/2<H<1$. Also we show that this estimator retains its properties if we replace $V_n^2(X,2)$ with $V_n^2(Y,2)$, where $Y(t)$ is the Milstein approximation of $X(t)$.

Keywords: fractional Brownian motion, quadratic variation, consistent estimator, Milstein approximation.

1. Introduction

In this paper we consider the stochastic integral equation

$$X_t=\xi+\int_0^t g(X_s)\,\mathrm{d}B_s^H,\quad t\in[0,1],\tag{1}$$

where B^H is a fractional Brownian motion (fBm) with the Hurst index $1/2<H<1$. The integral is Riemann–Stieltjes defined pathwise.

For $0\leqslant\alpha\leqslant1$, $\mathcal{C}^{1+\alpha}(\mathbb{R})$ denotes the set of all $\mathcal{C}^1$-functions $g\colon\mathbb{R}\to\mathbb{R}$ such that

$$|g'|_\infty+|g'|_\alpha:=\sup_x|g'(x)|+\sup_{x\neq y}\frac{|g'(x)-g'(y)|}{|x-y|^\alpha}<\infty.$$

Let $g\in\mathcal{C}^{1+\alpha}(\mathbb{R})$, $0\leqslant\alpha\leqslant1$. For $1\leqslant p\leqslant1+\alpha$ there exists a unique solution of the equation (1) with almost all sample paths in the class of all continuous functions defined on [0, 1] with bounded p-variation (see [2,3]).

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For a real-valued process $Y = (Y_t)$, $t \in [0, 1]$, we define the first and second order quadratic variations as

$$V_n(Y, 2) = \sum_{k=1}^{n} \big(\Delta_k Y\big)^2 \quad \text{and} \quad V_n^2(Y, 2) = \sum_{k=1}^{n-1} \big(\Delta_k^2 Y\big)^2,$$

where $\Delta_k Y = Y(t_k^n) - Y(t_{k-1}^n)$ and $\Delta_k^2 Y = Y(t_{k+1}^n) - 2Y(t_k^n) + Y(t_{k-1}^n)$, $t_k^n = \frac{k}{n}$. For simplicity we shall omit the index n for t in the sequel. The asymptotic behavior of these variations of Gaussian processes was considered in [1] (see also references in [1]). The first order quadratic variations for (1) was considered in [4].

The main result of this paper is the following theorem.

THEOREM 1. *Let $g \in \mathcal{C}^{1+\alpha}$, $0 < \alpha < 1$, and $|g|_\infty < \infty$. Then*

$$n^{2H-1} V_n^2(X, 2) \xrightarrow{\mathbf{a}.s.} \big(4 - 2^{2H}\big) \int_0^1 g^2(X(t))\,\mathrm{d}t.$$

It is easy to see that the following corollary holds.

CORROLORY 1. *Define*

$$\widehat{H}_n^{(2)} := \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}^2(X, 2)}{V_n^2(X, 2)}.$$

Then $\widehat{H}_n^{(2)} - H \xrightarrow{\mathbf{a}.s.} 0$ as $n \to \infty$.

In practice, however, we can only obtain "exact" sample paths if we explicitly know the solution of the considered stochastic integral equation which often is not true. In such cases we have no other choice but to replace these "exact" sample paths with their approximations (for example, the Euler or the Milstein ones).

Let $Y(t)$ be the Milstein approximation of the solution of the stochastic integral equation (1):

$$\begin{aligned} Y^n(t_k) &= Y^n(t_{k-1}) + g\big(Y^n(t_{k-1})\big) \cdot \Delta_k B^H \\ &\quad + \frac{1}{2} g\big(Y^n(t_{k-1})\big) g'\big(Y^n(t_{k-1})\big) \cdot \big(\Delta_k B^H\big)^2, \quad Y^n(t_0) = X(0) = \xi, \end{aligned}$$

where $\Delta_k B^H = B^H(t_k) - B^H(t_{k-1})$. We shall prove that if we replace the "exact" sample path with its Milstein approximation, the properties of the estimator $\widehat{H}_n$ will not change.

THEOREM 2. *Let*

$$\widehat{H}_n^{(1),M} := \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}(Y^n, 2)}{V_n(Y^n, 2)}.$$

and

$$\widehat{H}_n^{(2),M} := \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{V_{2n}^2(Y^n, 2)}{V_n^2(Y^n, 2)}.$$

Then $\widehat{H}_n^{(1),M} - H \xrightarrow{\mathbf{a}.s.} 0$ *and* $\widehat{H}_n^{(2),M} - H \xrightarrow{\mathbf{a}.s.} 0$ *as* $n \to \infty$.

2. Basic notions and auxiliary results

Let

$$\mathcal{W}_p([a,b]) := \{ f\colon [a,b] \to \mathbb{R}\colon v_p(f;[a,b]) < \infty \},$$

where

$$v_p(f;[a,b]) = \sup_{\varkappa} \sum_{k=1}^{n} \left| f(x_k) - f(x_{k-1}) \right|^p,$$

$\varkappa = \{x_i\colon i = 0, \dots, n\}$ being all finite partitions of $[a,b]$ such that $a = x_0 < x_i < \dots < x_n = b$.

Let $V_p(f) := V_p(f;[a,b]) = v_p^{1/p}(f;[a,b])$. $V_p(f)$ is a non-increasing function of p, that is, if $0 < q < p$ then $V_p(f) \leqslant V_q(f)$.

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$. It is known that

$$V_p\Bigl(\int_a^{\cdot} f\,\mathrm{d}h; [a,b]\Bigr) \leqslant C_{p,q} V_{q,\infty}\bigl(f;[a,b]\bigr) V_p\bigl(h;[a,b]\bigr), \tag{2}$$

where $V_{q,\infty}(f;[a,b]) = V_q(f;[a,b]) + \sup_{a \leqslant x \leqslant b} |f(x)|$, $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and $\zeta(s) = \sum_{n \geqslant 1} n^{-s}$.

The Young's version of Holder's inequality

$$\sum_{k=1}^{n} |a_k b_k| \leqslant \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q} \tag{3}$$

holds if $1/p + 1/q \geqslant 1$ for any $p, q > 0$.

Let $f \in \mathcal{W}_q([a,b])$ and $g \in \mathcal{W}_p([a,b])$. For any partition $\varkappa$ of $[a,b]$ and for $p^{-1} + q^{-1} \geqslant 1$ we have by Hölder inequality

$$\sum_i V_q\bigl(f;[x_{i-1},x_i]\bigr) V_p\bigl(g;[x_{i-1},x_i]\bigr) \leqslant V_q\bigl(f;[a,b]\bigr) V_p\bigl(g;[a,b]\bigr). \tag{4}$$

Since almost all sample paths of the B^H, $1/2 \leqslant H < 1$, are locally Hölder continuous, we have

$$V_p\bigl(B^H;[s,t]\bigr) \leqslant L_T^{H,1/p} (t-s)^{1/p}, \tag{5}$$

where $s < t \leqslant T,\ p > 1/H$,

$$L_T^{H,\gamma} = \sup_{\substack{s\neq t\\ s,t\leqslant T}} \frac{|B_t^H - B_s^H|}{|t-s|^\gamma}, \quad 0<\gamma<H, \qquad \mathbf{E}\big(L_T^{H,\gamma}\big)^k < \infty, \quad \forall k \geqslant 1.$$

3. Proofs

Proof of Theorem 1. From [4] it is easy to see that instead of evaluating $V_n^2(X,2)$ we can evaluate

$$\begin{aligned}
&\sum_{k=1}^{n-1} \big\{ g^2(X(t_k))\big(\Delta_{k+1}B^H\big)^2 + g^2(X(t_{k-1}))\big(\Delta_k B^H\big)^2 \\
&\qquad - 2g(X(t_k))g(X(t_{k-1}))\Delta_{k+1}B^H \Delta_k B^H \big\} \\
&\quad = \sum_{k=1}^{n-1} \big\{ g^2(X(t_k))\big(\Delta_k^2 B^H\big)^2 \\
&\qquad - \big[g^2(X(t_k)) - g^2(X(t_{k-1}))\big]\big[\big(\Delta_k B^H\big)^2 - \Delta_{k+1}B^H\Delta_k B^H\big] \\
&\qquad + \big[g(X(t_k)) - g(X(t_{k-1}))\big]^2 \Delta_{k+1}B^H\Delta_k B^H \big\}.
\end{aligned}$$

Further by (3) and (4) for $p > 1$ we get

$$\begin{aligned}
&\sum_{k=1}^{n-1} \big[g(X(t_k)) - g(X(t_{k-1}))\big]^2 \big|\Delta_{k+1}B^H\Delta_k B^H\big| \\
&\qquad \leqslant \max_{1\leqslant k\leqslant n} \big[g(X(t_k)) - g(X(t_{k-1}))\big]^2 V_p^2\big(B^H;[0,1]\big)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{n-1} \big|g^2(X(t_k)) - g^2(X(t_{k-1}))\big| \cdot \big|\big(\Delta_k B^H\big)^2 - \Delta_{k+1}B^H\Delta_k B^H\big| \\
&\quad \leqslant 2|g|_\infty \max_{1\leqslant k\leqslant n} \big|g(X(t_k)) - g(X(t_{k-1}))\big| \\
&\qquad \times \bigg\{ \max_{1\leqslant k\leqslant n-1} |\Delta_k B^H|^{2-p} v_p\big(B^H;[0,1]\big) + V_p^2\big(B^H;[0,1]\big) \bigg\}.
\end{aligned}$$

Also, by (5) and (2) we get

$$\max_{1\leqslant k\leqslant n} \big|g(X(t_k)) - g(X(t_{k-1}))\big| \leqslant 2n^{-1/p} C_{p,q/\alpha} |g'|_\infty |g|_\alpha L_1^{H,1/p} V_q^\alpha\big(X;[0,1]\big).$$

Therefore

$$n^{2H-1}\sum_{k=1}^{n-1}\Big|\big[g^2(X(t_k))-g^2(X(t_{k-1}))\big]\big[\big(\Delta_k B^H\big)^2-\Delta_{k+1}B^H\Delta_k B^H\big] +\big[g(X(t_k))-g(X(t_{k-1}))\big]^2\Delta_{k+1}B^H\Delta_k B^H\Big| \xrightarrow{\text{a.s.}} 0,$$

as $n\to\infty$. To this end it suffices to take p sufficiently close to H and such that $2H<1+1/p^2$.

Consequently, the theorem will be proved if

$$n^{2H-1}\sum_{k=1}^{n-1} g^2(X(t_k))\big(\Delta_k^2 B^H\big)^2 \xrightarrow{\text{a.s.}} \big(4-2^{2H}\big)\int_0^1 g^2(X(t))\,\mathrm{d}t.$$

This follows from the Helly–Bray theorem.

Proof of Theorem 2. From [4] it follows that

$$n^{2H-1}\Big|V_n(X,2)-\sum_{k=1}^{n} g^2(X(t_{k-1}))\cdot\big(\Delta_k B^H\big)^2\Big| \xrightarrow{\text{a.s.}} 0 \quad \text{as } n\to\infty.$$

Consider the first order difference of $Y(t)$

$$\Delta_k Y = g(Y(t_{k-1})\Delta_k B^H + \frac{1}{2} g\big(Y(t_{k-1})\big)g'\big(Y(t_{k-1})\big)\big(\Delta_k B^H\big)^2.$$

Note that

$$\begin{aligned}
&n^{2H-1}\Big|V_n(Y,2)-\sum_{k=1}^{n} g^2\big(X(t_{k-1})\big)\big(\Delta_k B^H\big)^2\Big|\\
&\leqslant n^{2H-1}\Big|\sum_{k=1}^{n}\big[g^2\big(Y(t_{k-1})\big)-g^2\big(X(t_{k-1})\big)\big]\big(\Delta_k B^H\big)^2\Big|\\
&\quad+n^{2H-1}\Big|\sum_{k=1}^{n} g^2\big(Y(t_{k-1})\big)g'\big(Y(t_{k-1})\big)\big(\Delta_k B^H\big)^3\Big|\\
&\quad+\frac{1}{4}n^{2H-1}\Big|\sum_{k=1}^{n} g^2\big(Y(t_{k-1})\big)\Big(g'\big(Y(t_{k-1})\big)\Big)^2\big(\Delta_k B^H\big)^4\Big|\\
&\leqslant 2|g|_\infty \max_{1\leqslant k\leqslant n-1}\big|Y(t_k)-X(t_k)\big| n^{2H-1}\sum_{k=1}^{n}\big(\Delta_k B^H\big)^2
\end{aligned}$$

$$+ |g|_\infty^2 \cdot |g'|_\infty n^{-1/p} L_1^{H,1/p} n^{2H-1} \sum_{k=1}^{n} \big(\Delta_k B^H\big)^2 +$$

$$+ \frac{1}{4}|g|_\infty^2 \cdot |g'|_\infty^2 n^{-1/p^2} \big(L_1^{H,1/p}\big)^2 n^{2H-1} \sum_{k=1}^{n} \big(\Delta_k B^H\big)^2.$$

It is known that

$$\sup_{0\leqslant t\leqslant 1} \mathbb{E}\big|Y(t) - X(t)\big| = \mathcal{O}\bigg(\frac{1}{n^H}\bigg) \quad \text{as } n \to \infty.$$

This, together with the results of Theorem 1, implies that

$$n^{2H-1}\Bigg|\sum_{k=1}^{n}\big[V_n(Y,2) - g^2(X(t_{k-1}))\cdot\big(\Delta_k B^H\big)^2\big]\Bigg| \xrightarrow{\mathbf{a}.s.} 0 \quad \text{as } n \to \infty.$$

Consequently, this yields that

$$\widehat{H}_n^{(1),M} := \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{V_{2n}(Y,2)}{V_n(Y,2)}$$

also is a strongly consistent estimator of the Hurst index H. In a similiar way it is not hard to check that $\widehat{H}_n^{(2),M} - H \xrightarrow{\mathbf{a}.s.} 0$ as $n \to \infty$ as well.

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REZIUMĖ

K. Kubilius, D. Melichov. Stochastinės integralinės lygties sprendinio Hursto indekso vertinimas

Tarkime, $X(t)$ yra stochastinės diferencialinės lygties, valdomos trupmeninio Brauno judesio B^H, sprendinys, o $V_n^2(X,2) = \sum_{k=1}^{n-1}(\Delta_k^2 X)^2$ yra antrosios eilės kvadratinė variacija, čia $\Delta_k^2 X = X\big(\frac{k+1}{n}\big) - 2X\big(\frac{k}{n}\big) + X\big(\frac{k-1}{n}\big)$. Rastos sąlygos kada $n^{2H-1}V_n^2(X,2)$ konverguoja b.v., kai $n \to \infty$. Tai leidžia gauti stipriai pagrįstą Hursto indekso H, $1/2 < H < 1$, įvertinį. Įrodoma, kad įverčio savybės nepasikeis, jei $V_n^2(X,2)$ pakeisime $V_n^2(Y,2)$, čia $Y(t)$ – proceso $X(t)$ Milšteino aproksimacija.

Raktiniai žodžiai: trupmeninis Brauno judesys, kvadratinė variacija, pagrįstas įvertis, Milšteino aproksimacija.