

# On limit theorems for random fields

Rimas BANYS

Vilnius Gediminas Technical University  
Saulėtekio 11, 10223 Vilnius, Lithuania  
e-mail: rimasbanys@takas.lt

**Abstract.** A complete separable metric space of functions defined on the positive quadrant of the plane is constructed. The characteristic property of these functions is that at every point  $x$  there exist two lines intersecting at this point such that limits  $\lim_{y \rightarrow x} f(y)$  exist when  $y$  approaches  $x$  along any path not intersecting these lines. A criterion of compactness of subsets of this space is obtained.

*Keywords:* Skorokhod topology, weak convergence, tightness, random fields.

## 1. Introduction

The theory of weak convergence in the function space  $D[0, 1]$  (Billingsley [3]) has been extended by Bickel and Wichura [2], Neuhaus [6], and Straf [7] to  $D[0, 1]^q$ , by Lindvall [5] to  $D[0, \infty)$ , and by Ivanoff [4] to  $D[0, \infty)^q$ . In case  $q = 2$  the spaces  $D[0, 1]^2$  and  $D[0, \infty)^2$  consist of functions with two parameters having so called quadrant limits. Discontinuity points of these functions form lines parallel to the coordinate axes. Here we construct a more general space of functions with two parameters having so called net limits. Discontinuity points of these functions form lines not necessarily parallel to coordinate axes. The restriction of this space to the space of functions defined on the unit square is a particular case of the space introduced by Banys and Surgailis [1].

The functions under consideration have the property that at each point  $x$  in the domain of definition there exist two lines intersecting at  $x$  such that limits  $\lim_{y \rightarrow x} f(y)$  exist when  $y$  approaches  $x$  along any path not intersecting these lines.

## 2. Definitions

Let  $R^2$  be the two-dimensional Euclidean space with the norm  $\|x\| = \sqrt{x_1^2 + x_2^2}$  of an element  $x = (x_1, x_2) \in R^2$ . We shall denote by  $\bar{U}$  and  $\partial U$  respectively the closure and the boundary of a set  $U \subset R^2$  and by  $B_\delta(x)$  the open circle with radius  $\delta$  and center  $x$ .

**DEFINITION 1.** A union  $\Phi = \varphi_1 \cup \varphi_2$  of line segments  $\varphi_1$  and  $\varphi_2$  passing through a point  $x \in R^2$  is called an elementary net at  $x$  if one of the following two conditions holds: (i)  $\Phi$  is a line segment itself and  $x$  is an interior point of  $\Phi$ ; (ii)  $\varphi_1$  and  $\varphi_2$  are not aligned with a straight line.

Let  $X = [0, 1]^2 = \{x = (x_1, x_2), 0 \leq x_1, x_2 \leq 1\}$  be the unit square. Denote by  $\partial_i X$ ,  $i = 1, \dots, 4$ , the sides of  $X$  and by  $\partial_0 X$  the set of its corner points.

DEFINITION 2. A finite union  $\Phi = \bigcup_{i=1}^n \varphi_i$  ( $n \geq 4$ ) of line segments lying in  $X$  is called a net if  $\partial X \subset \Phi$  and for every  $x \in \Phi$  there exists a circle  $B = B(x)$  such that  $\Phi \cap \overline{B}$  is an elementary net at  $x$ .

If  $\Phi \cap \overline{B(x)}$  is formed by two line segments not aligned with a straight line, then  $x$  is called a node point of the net  $\Phi$ .

Let  $\Delta = \{A_1, \dots, A_n\}$  be a finite partition of  $X$ , i.e.  $X = \bigcup_{i=1}^n A_i$  and  $A_i \cap A_j = \emptyset$  if  $i \neq j$ . Denote  $\partial \Delta = \bigcup_{i=1}^n \partial A_i$ . Let  $\Psi_X$  be the collection of all finite partitions  $\Delta$  of  $X$  such that  $\partial \Delta$  is a net.

DEFINITION 3. A function  $f : X \rightarrow R^1$  is simple if

$$f = \sum_{i=1}^m r_i \chi_{A_i} \quad (m \geq 1),$$

where  $\chi_A$  is indicator function,  $\Delta = \{A_1, \dots, A_m\} \in \Psi_X$  and, moreover,

$$f(x) = \limsup_{y \rightarrow x} f(y), \quad x \in X. \quad (1)$$

Given simple function  $f = \sum_{i=1}^m r_i \chi_{A_i}$ , for each  $x \in \partial \Delta$  there exists a circle  $B(x)$  such that  $\partial \Delta \cap \overline{B(x)}$  is the elementary net at  $x$  and limits

$$\{ \lim f(y), y \rightarrow x, y \in A_i \cap B(x) \} \quad (2)$$

exist for such  $i$  for which  $x$  is accumulation point of  $A_i$ . (The number of such  $A_i$  is no more than 4).

The class of simple functions will be denoted by  $D_0$ .

### 3. The space $D[0, 1]^2$

Denote by  $D(X)$  the uniform closure of  $D_0$  in the space of all bounded functions from  $X$  to  $R^1$ .

Let  $f \in D(X)$  and  $\Phi$  be an elementary net at  $x \in X$ . The elementary net  $\Phi$  divides any sufficiently small circle  $B(x)$  into disjoint components  $B^1, \dots, B^n$  ( $2 \leq n \leq 4$ ). We say that  $f$  has  $\Phi$ -limits at  $x$  if there exist limits

$$\lim_{y \rightarrow x, y \in B^i} f(y) = f^{(i)}(x), \quad i = 1, \dots, n. \quad (3)$$

Put  $L(x, f) = \{f^{(i)}(x), i = 1, \dots, n\}$ . It follows that  $f(x) = \max_{1 \leq i \leq n} f^{(i)}(x)$  and limits  $\lim_{y \rightarrow x} f(y)$  exist when  $y$  approaches  $x$  along the lines forming the elementary net  $\Phi$ , and these limits are in  $L(x, f)$ . Therefore,

$$f(x) = \limsup_{y \rightarrow x} f(y), \quad x \in X.$$

Now we define a modulus which plays in  $D$  the same role as the similar modulus does in  $D[0, 1]$  and in  $D[0, 1]^2$ . Let  $\Phi$  be a net. We shall use the following notations:  $S(\Phi) = \min\{\|x - y\|, x \neq y\}$ , where the minimum extends over the node points of  $\Phi$ ;  $S_\Phi(x, y)$  – the smallest length of all line segments  $\varphi_i \subset \Phi$  union  $\cup \varphi_i$  of which connects the points  $x$  and  $y$ . Put  $S_\Phi(x, y) = \sqrt{2}$  if  $x$  and  $y$  are not connected by any part of  $\Phi$ ;

$$R(\Phi) = \inf_{x \neq y} \left\{ \frac{\sqrt{2}\|x - y\|}{S_\Phi(x, y)}, x, y \in \Phi \right\};$$

$\Gamma(\Phi) = \min_x \sin \gamma_x$ , where the minimum extends over all node points  $x$  of  $\Phi$ , and  $\gamma_x$  is the smallest positive angle between two line segments  $\varphi_1, \varphi_2 \subset \Phi$  intersecting at the node point  $x$ . For an arbitrary net  $\Phi$  and the boundary  $\partial X$  the following relations hold:

$$0 < S(\Phi), R(\Phi), \Gamma(\Phi) \leq 1, \quad S(\partial X) = R(\partial X) = \Gamma(\partial X) = 1.$$

With a net  $\Phi$ , we associate a modulus of its “smoothness” defined by

$$\kappa(\Phi) = S(\Phi)R(\Phi)\Gamma(\Phi).$$

Obviously,  $0 < \kappa(\Phi) \leq 1$  and  $\kappa(\partial X) = 1$ .

Let  $f \in D(X)$ ,  $\Delta = \{A_1, \dots, A_n\} \in \Psi_X$ , and  $\delta > 0$  be given. Put  $\omega_f(\Delta) = \max_{1 \leq i \leq n} \omega_f(A_i)$ , where  $\omega_f(A_i) = \sup\{|f(x) - f(y)|, x, y \in A_i, i = 1, \dots, n$ , and  $w_\delta(f) = \inf\{\omega_f(\Delta), \Delta \in \Psi_X, \kappa(\partial \Delta) > \delta\}$ , where the minimum extends over all the partitions  $\Delta \in \Psi_X$  satisfying  $\kappa(\partial \Delta) > \delta$ . To introduce a metric on  $D(X)$  we begin with a group  $\Lambda$  of one to one mappings of  $X$  onto itself. A one to one mapping  $\lambda$  of  $X$  onto itself is called a  $C^1$  – diffeomorphism if it is continuous together with its inverse  $\lambda^{-1}$  and the partial derivatives

$$\frac{\partial \lambda_i}{\partial x_j}, \frac{\partial \lambda_i^{-1}}{\partial x_j}, \quad i, j = 1, 2,$$

where  $x = (x_1, x_2)$  and  $\lambda(x) = (\lambda_1(x), \lambda_2(x))$ .

Let  $\Lambda$  be the class of all  $C^1$  – diffeomorphisms such that  $\lambda(x) = x$ ,  $x \in \partial_0 X$ , and  $\lambda(\partial_i X) = \partial_i X$ ,  $i = 1, \dots, 4$ . Denote by  $C_\lambda$  the matrix of the partial derivatives of  $\lambda$ , i.e.,

$$C_\lambda(x) = \left[ \frac{\partial \lambda_i(x)}{\partial x_j} \right]_{i,j=1,2}.$$

Given matrix  $A = [a_{i,j}]$ , its norm is defined by  $|A| = 2 \max_{i,j} |a_{i,j}|$ . Let  $I$  be the identity matrix. Define function  $|\cdot|: \Lambda \rightarrow R^1$  by

$$|\lambda| = \sup_x F(|C_\lambda(x) - I|) + \sup_x F(|C_{\lambda^{-1}}(x) - I|),$$

where  $F(t) = t(1+t)^{-1}$ ,  $t \geq 0$ . We define a metric in  $D(X)$  by  $d(f, g) = \inf_\varepsilon \{\varepsilon > 0: \text{there exists } \lambda \in \Lambda \text{ such that } |\lambda| < \varepsilon \text{ and } \sup_x |f(x) - g(\lambda(x))| < \varepsilon\}$ . The space  $(D, d)$  is a complete separable metric space.

Now we formulate three statements on the compactness and tightness in  $D$  which are analogous to Theorems 1–3 in [1].

**THEOREM 1.** *A set  $A \subset D$  has a compact closure in  $(D, d)$  if and only if*

$$\sup_{f \in A} \sup_x |f(x)| < \infty \quad \text{and} \quad \lim_{\delta \rightarrow 0} \sup_{f \in A} w_\delta(f) = 0.$$

Denote  $\mathcal{B}(D)$  the  $\sigma$ -algebra of Borel subsets of  $(D, d)$ .

**THEOREM 2.** *The sequence  $\{P_n\}$  of probability measures on  $(D, \mathcal{B}(D))$  is tight if and only if the following conditions hold:*

$$\lim_{a \rightarrow \infty} \limsup_{n \rightarrow \infty} P_n \left\{ f: \sup_x |f(x)| > a \right\} = 0$$

and for each  $\varepsilon > 0$

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P_n \{f: w_\delta(f) \geq \varepsilon\} = 0.$$

**THEOREM 3.** *The sequence  $\xi_n$  of random fields converges in distribution to the random field  $\xi$  in  $(D, d)$  if and only if the following conditions hold:*

$$\xi_n \xrightarrow{fdd} \xi$$

and for all  $\varepsilon > 0$

$$\lim_{\delta \rightarrow 0} \limsup P \{w_\delta(\xi_n) > \varepsilon\} = 0.$$

#### 4. The space $D[0, \infty)^2$

Now we construct a similar space of functions defined on  $R_+^2 = [0, \infty)^2$  extending the techniques used by Lindvall [5].

Let  $D^c[0, 1]^2 = \{f \in D[0, 1]^2: f \text{ is continuous at } x = (x_1, x_2) \text{ if } x_1 = 1 \text{ or } x_2 = 1\}$ . Denote by  $D^c[0, 1)^2$  the space of functions from  $D^c[0, 1]^2$  restricted to  $[0, 1)^2$ . In  $D^c[0, 1)^2$ , define metric  $d'$  by

$$d'(f, g) = d(\bar{f}, \bar{g}), \quad f, g \in D^c[0, 1)^2,$$

where  $\bar{f}, \bar{g} \in D^c[0, 1]^2$ ,  $\bar{f}(x) = f(x)$ ,  $\bar{g}(x) = g(x)$  for  $x \in [0, 1)^2$ . The space  $D^c[0, 1)^2$  is closed in  $D[0, 1)^2$  and, therefore, is a complete separable metric space. Let  $R_+^2 = \{(x_1, x_2) \in R^2, 0 \leq x_i < \infty, i = 1, 2\}$ . The function space  $D(R_+^2)$  is defined analogously to  $D(X)$  with minor changes. Denote  $D^c(R_+^2) = \{f \in D(R_+^2): \text{limits } \lim_{x \rightarrow x^*} f(x) = f(x^*) \text{ exist and } f(x^*) < \infty \text{ for } x^* = (x_1^*, x_2^*), x_1^* = \infty \text{ or } x_2^* = \infty\}$ . For  $x = (x_1, x_2) \in [0, 1)^2$ , put  $\varphi(x) = (-\log(1 - x_1), -\log(1 - x_2))$  and define the mapping  $\Phi: D^c(R_+^2) \rightarrow D^c[0, 1)^2$  by

$$\Phi(f) = f \circ \varphi, \quad f \in D^c(R_+^2).$$

The mapping  $\Phi$  is a bijection, and  $D^c(R_+^2)$  with metric  $h(f, g) = d'(\Phi(f), \Phi(g))$ ,  $f, g \in D^c(R_+^2)$  is a complete separable metric space. Let

$$D_\infty^c = \{\langle f_k \rangle_{k=1}^\infty, f_k \in D^c(R_+^2)\}.$$

On  $D_\infty^c$  a metric  $\rho$  is defined by

$$\rho(\xi, \eta) = \sum_{k=1}^{\infty} \frac{h(f_k, g_k)}{1 + h(f_k, g_k)} 2^{-k},$$

where  $\xi = \langle f_k \rangle_{k=1}^\infty$ ,  $\eta = \langle g_k \rangle_{k=1}^\infty$ .  $(D_\infty^c, \rho)$  is again a complete and separable. For  $k = 1, 2, \dots$  put  $g_k(x) = g'_k(x_1)g'_k(x_2)$ , where  $x = (x_1, x_2) \in R_+^2$ , and  $g'_k$  are defined by  $g'_k(t) = 1$  for  $t \leq k$ ,  $g'_k(t) = k + 1 - t$  for  $k < t < k + 1$ , and  $g'_k(t) = 0$  for  $t \geq k + 1$ .

Define the mappings  $c_k: D(R_+^2) \rightarrow D^c(R_+^2)$  by  $c_k(f) = f \times g_k$ , and the mapping  $\Psi: D(R_+^2) \rightarrow D_\infty^c$  by  $\Psi(f) = \langle c_k(f) \rangle_{k=1}^\infty$ .

Now we can define a metric on  $D(R_+^2)$ . For  $f, g \in D(R_+^2)$  put

$$d_\infty(f, g) = \rho(\Psi(f), \Psi(g)).$$

**THEOREM 4.** *The space  $D(R_+^2)$  endowed with metric  $d_\infty$  is a complete separable metric space.*

Proof follows from the fact that  $\Psi(D(R_+^2))$  is closed in  $D_\infty^c$ .

**THEOREM 5.** *Let  $f, f_1, f_2, \dots \in D(R_+^2)$ . Then  $f_n \rightarrow f$  if and only if there exist a sequence  $\langle \lambda_n \rangle_{n=1}^\infty$ ,  $\lambda_n \in \Lambda(R_+^2)$ , such that for all  $a > 0$  the following relations hold:*

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, a]^2} |f_n(\lambda_n(x)) - f(x)| = 0, \quad \lim_{n \rightarrow \infty} \sup_{x \in [0, a]^2} \left| \frac{\partial \lambda_n^i}{\partial x_j}(x) - \delta_{ij} \right| = 0,$$

and

$$\lim_{n \rightarrow \infty} \sup_{x \in [0, a]^2} \left| \frac{\partial (\lambda_n^{-1})^i}{\partial x_j}(x) - \delta_{ij} \right| = 0,$$

where  $x = (x_1, x_2)$ ,  $\lambda_n = (\lambda_n^1, \lambda_n^2)$ ,  $\lambda_n^{-1} = ((\lambda_n^{-1})^1, (\lambda_n^{-1})^2)$ ,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The theorem can be proved using similar arguments like in the proof of Theorem 3.1 in [4].

Given  $x_1, \dots, x_k \in R_+^2$  define the projection  $\pi_{x_1, \dots, x_k}: D(R_+^2) \rightarrow R^k$  as usual:

$$\pi_{x_1, \dots, x_k}(f) = (f(x_1), \dots, f(x_k)), \quad f \in D(R_+^2).$$

Let  $T \subset R_+^2$ . Denote by  $\mathcal{F}_T$  the  $\sigma$ -algebra containing all the finite-dimensional sets  $\pi_{x_1, \dots, x_k}^{-1} A$ , where  $A$  are Borel subsets of  $R^k$ ,  $x_1, \dots, x_k \in T$ ,  $k = 1, 2, \dots$

**THEOREM 6.** *If  $T$  is dense in  $R_+^2$ , then  $\mathcal{F}_T$  coincides with Borel  $\sigma$ -algebra of subsets of  $D(R_+^2)$ .*

*Proof.* Let  $\pi_{x_1, \dots, x_k}^c$  be a restriction of  $\pi_{x_1, \dots, x_k}$  in  $D^c(R_+^2)$ . Projections  $\pi_{x_1, \dots, x_k}^c$  are measurable. If  $n > \max |x_i|$ , where  $|x_i| = \max \{x_i^1, x_i^2\}$ , then  $\pi_{x_1, \dots, x_k} = \pi_{x_1, \dots, x_k}^c \circ c_n$ . Therefore,  $\pi_{x_1, \dots, x_k}$  are measurable as well. Thus  $\mathcal{F}_T$  is contained in the Borel  $\sigma$ -algebra of subsets of  $D(R_+^2)$ . The opposite inclusion follows from Proposition 1.4 in [8].

For  $a = (a_1, a_2) \in R^2$ , denote  $X_a = \{(x_1, x_2): 0 \leq x_i \leq a_i, i = 1, 2\}$ . Let  $D(X_a)$  be the space of functions from  $X_a$  to  $R^1$  defined analogously to the space  $D(X)$ . Define the mapping  $r_a: D(R_+^2) \rightarrow D(X_a)$  by  $r_a f(x) = f(x)$ ,  $x \in X_a$ . For a probability measure  $P$  on  $D(R_+^2)$ , let  $T_P$  be the set of those  $a \in R_+^2$  for which

$$P\{f: r_a f \in D(X_a) \text{ and } r_a \text{ is continuous at } f\} = 1.$$

**THEOREM 7.** *Let  $P, P_1, P_2, \dots$  be probability measures on  $D(R_+^2)$ . Then the sequence  $\{P_n, n = 1, 2, \dots\}$  converges weakly to  $P$  if and only if for all  $a \in T_P$  the sequence  $\{P_n r_a^{-1}, n = 1, 2, \dots\}$  converges weakly to  $P r_a^{-1}$ .*

The proof is analogous to the proof of Theorem 3 in [5].

### References

1. R. Banys and D. Surgailis. On weak convergence of random fields. *Lith. Math. J.*, 39:134–145, 1999.
2. P.J. Bickel and M.J. Wichura. Convergence criteria for multiparameter stochastic processes and some applications. *Ann. Math. Statist.*, 42:1656–1670, 1971.
3. P. Billingsley. *Convergence of Probability Measures*. John Wiley, New York, 1968.
4. B. Gail Ivanoff. The function space  $D([0, \infty)^q, E)$ . *The Canadian Journal of Statistics*, 8:179–191, 1980.
5. T. Lindvall. Weak convergence of probability measures and random functions in the function space  $D[0, \infty)$ . *J. Appl. Probab.*, 10:109–121, 1973.
6. G. Neuhaus. On weak convergence of stochastic processes with multidimensional time parameter. *Ann. Math. Statist.*, 42:1285–1296, 1971.
7. M.L. Straf. Weak convergence of stochastic processes with several parameters. In: *Proc. 6th Berkeley Symp. Math. Statist. Probab.*, Vol. 2, 187–221, 1972.
8. N. Vakhania, V. Tarieladze and S. Chobonian. *Probability Distributions in Banach Spaces*. Nauka, Moscow, 1985 (in Russian).

### REZIUMÉ

#### **R. Banys. Apie atsitiktinių laukų ribines teoremas**

Sukonstruota funkcijų, apibrėžtų plokštumos pirmajame kvadrante, pilnoji separabelioji metrinė erdvė. Šios erdvės funkcijų charakteringoji savybė yra tai, kad kiekviename taške  $x$  galima nubrėžti dvi tieses taip, kad ribos  $\lim_{y \rightarrow x} f(y)$  egzistuotų, kai  $y$  artėja prie  $x$  bet kuria trajektorija, nekertančia šių tiesių. Pateiktas šios erdvės poaibių kompaktiškumo kriterijus.

*Raktiniai žodžiai:* Skorochodo topologija, silpnasis konvergavimas, tankumas, atsitiktiniai laukai.