

Cut free sequent calculus for logic $S5_n(ED)$

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Abstract. Hilbert style, Gentzen style sequent and Kanger style sequent calculi for logic $S5_n(ED)$ are considered in this paper. Gentzen style sequent calculus is constructed and its equivalence with Hilbert style system is proved, getting soundness and completeness of Gentzen style system. Kanger style indexed sequent calculus is defined for cut elimination.

Keywords: epistemic logic, distributed knowledge, knowledge of everybody, sequent calculus, cut elimination.

Introduction

$S5_n(ED)$ is one of the epistemic logics. Epistemic logic is the logic of knowledge and belief. It provides insight into the properties of individual agents and means to model complicated scenarios involving groups of agents. In addition to its relevance for traditional philosophical problems, epistemic logic has many applications in computer science and artificial intelligence. It can be used for distributed systems, knowledge base merging, robotics or network security.

Applying logic to computer science, it is important to have mechanical proof system of the logic. Suggestions of proof theory of epistemic logic are usually limited to Hilbert style axiomatizations [2]. Several works were done by Regimantas Pliuškevičius and Aida Pliuškevičienė in [7], and Raul Hakli and Sara Negri in [2], proposing cut free Gentzen style sequent calculi for $S4_n(D)$ and $S5_n(D)$. Aida Pliuškevičienė in [6] presented indexed sequent calculus, which is known as Kanger style, for logic $S5_2$. Our work is strongly related to [6] and describes Kanger style sequent calculus for logic $S5_n(ED)$.

Firstly, we define syntax of $S5_n(ED)$ in Section 1, then prove equivalence between Hilbert style calculus $HS-S5_n(ED)$ and Gentzen style sequent calculus $GS-S5_n(ED)$ in Sections 2 and 3, to get soundness and completeness of $GS-S5_n(ED)$. Finally, cut free Kanger style sequent calculus is presented in Section 4.

1 Syntax of logic $S5_n(ED)$

The language of $S5_n(ED)$ contains:

- Symbols of atomic propositions: $p_1, p_2, p_3, q_1, q_2, q_3, \dots$;
- Symbols of formulas: $A_1, A_2, A_3, B_1, B_2, \dots$;
- Logical connectives: $\wedge, \vee, \neg, \rightarrow$;

- Constants of agents: $i_1, i_2, i_3, \dots, i_n$;
- Operators: $E, D, K_1, K_2, \dots, K_n$.

Formula $K_i A$ with knowledge operator K_i means “agent i knows A ”. Operator E signifies “everybody knows” and EA is defined as the conjunction $K_1 A \wedge \dots \wedge K_n A$, expressing “everybody knows that A ”. Distributed knowledge A of within a group of agents is signed DA and means that A follows from what the members of group of agents individually know. For instance, having the group of three agents of which the first one knows B , the second one knows $B \rightarrow C$, and the third one knows $B \wedge C \rightarrow A$, A is distributed knowledge in this group.

2 Hilbert style calculus for logic $S5_n(ED)$

W. van der Hoek and J.J.-Ch. Meyer defined Hilbert style calculus for logic $S5_n(EDC)$ in [8]. We'll use the subsystem of this calculus, selecting axioms and rules related to operators K_i , E and D :

- Any axiomatization for propositional logic;
- Axioms for operator K_i :

$$\mathbf{K1.} (K_i A \wedge K_i (A \rightarrow B)) \rightarrow K_i B,$$

$$\mathbf{K2.} K_i A \rightarrow A,$$

$$\mathbf{K3.} K_i A \rightarrow K_i K_i A,$$

$$\mathbf{K4.} \neg K_i A \rightarrow K_i \neg K_i A;$$

- Axioms for operator E :

$$\mathbf{E1.} (K_1 A \wedge \dots \wedge K_n A) \rightarrow EA,$$

$$\mathbf{E2.} EA \rightarrow (K_1 A \wedge \dots \wedge K_n A);$$

- Axioms for operator D :

$$\mathbf{D1.} K_i A \rightarrow DA,$$

$$\mathbf{D2.} (DA \wedge D(A \rightarrow B)) \rightarrow DB,$$

$$\mathbf{D3.} DA \rightarrow A,$$

$$\mathbf{D4.} DA \rightarrow DDA,$$

$$\mathbf{D5.} \neg DA \rightarrow D\neg DA;$$

- Rules:

$$\frac{A, A \rightarrow B}{B} \text{ (R1)} \quad \frac{A}{K_i A} \text{ (R2)} \quad \frac{A}{DA} \text{ (R3)}.$$

W. van der Hoek and J.J.-Ch. Meyer have proved that $HS-S5_n(EDC)$ system is sound and complete. Therefore we get soundness and completeness for $HS-S5_n(ED)$.

3 Gentzen style sequent calculus for logic $S5_n(\text{ED})$

Regimantas Pliuškevičius and Aida Pliuškevičienė considered Gentzen style sequent calculus for $S4_n(\text{D})$ in [7]. Masao Ohnishi and Kazuo Matsumoto suggested Gentzen style sequent system for $S5_n$ in [4] and [5]. We'll use these ideas to construct Gentzen style sequent system $\text{GS-S5}_n(\text{ED})$ for logic $S5_n(\text{ED})$:

- Axiom: $\Gamma, A \vdash \Delta, A$;
- Inference rules of the sequent calculus G for propositional logic [1] (Jean H. Gallier, 2003);
- Inference rules for operators K_i, E, D :

$$\frac{\Gamma, A, K_i A \vdash \Delta}{\Gamma, K_i A \vdash \Delta} (K_i \vdash), \quad \frac{K_i \Gamma_2 \vdash K_i \Delta_2, A}{\Gamma_1, K_i \Gamma_2 \vdash K_i \Delta_2, K_i A, \Delta_1} (\vdash K_i),$$

$$\frac{\Gamma, K_1 A \wedge \dots \wedge K_n A \vdash \Delta}{\Gamma, EA \vdash \Delta} (E \vdash), \quad \frac{\Gamma \vdash K_1 A \wedge \dots \wedge K_n A, \Delta}{\Gamma \vdash EA, \Delta} (\vdash E),$$

$$\frac{\Gamma, A, DA \vdash \Delta}{\Gamma, DA \vdash \Delta} (D \vdash), \quad \frac{D \Gamma_2 \vdash D \Delta_2, A}{\Gamma_1, D \Gamma_2 \vdash D \Delta_2, DA, \Delta_1} (\vdash D),$$

$$\frac{\Gamma_2 \vdash A}{\Gamma_1, K \Gamma_2 \vdash \Delta, DA} (\text{Interaction});$$

- Structural rules and cut rule:

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} (\text{Weakening } \vdash), \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} (\vdash \text{Weakening}),$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} (\text{Contraction } \vdash), \quad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} (\vdash \text{Contraction}),$$

$$\frac{\Gamma_1 \vdash \Delta_1, A \quad A, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} (\text{Cut})$$

where Γ and Δ are multisets of formulas. $K_i \Gamma_2$ and $K_i \Delta_2$ are empty sets or consists of formulas $K_i A_1, \dots, K_i A_n$ ($n \geq 1$). Analogously, $D \Gamma_2$ and $D \Delta_2$ are empty sets or DA_1, \dots, DA_n ($n \geq 1$). $K \Gamma$ means empty set or multiset of formulas $K_1 A_{1,1}, \dots, K_1 A_{1,l_1}, \dots, K_n A_{n,1}, \dots, K_n A_{n,l_n}$ ($n \geq 1$).

To obtain soundness and completeness for system $\text{GS-S5}_n(\text{ED})$, the equivalence between $\text{GS-S5}_n(\text{ED})$ and $\text{HS-S5}_n(\text{ED})$ must be proved.

Definition 1 [Formula of a sequent]. If S is a sequent $A_1, \dots, A_l \vdash B_1, \dots, B_k$ then *formula of a sequent* is $F(S) = (A_1 \wedge \dots \wedge A_l) \rightarrow (B_1 \vee \dots \vee B_k)$.

Lemma 1 [Equivalence between $\text{GS-S5}_n(\text{ED})$ and $\text{HS-S5}_n(\text{ED})$]. *Sequent S is provable in $\text{GS-S5}_n(\text{ED})$ if and only if $F(S)$ is provable in $\text{HS-S5}_n(\text{ED})$.*

Proof. “ \Rightarrow ”. We'll use induction on number of steps N of proof of $F(S)$ in $\text{HS-S5}_n(\text{ED})$. Step is defined as usage of axiom or rule.

- Case $N = 1$: One of the axiom was used.
 - Let axiom K3 was used. Then its proof in $\text{GS-}S5_n(\text{ED})$ is:

$$\frac{\frac{K_i A \vdash K_i A}{K_i A \vdash K_i K_i A} (\vdash K_i)}{\vdash K_i A \rightarrow K_i K_i A} (\vdash \rightarrow)$$

- Other axioms are proved analogously.
- Case $N > 1$: One of the rule R1, R2 or R3 was applied on the last step.
 - Let rule R2 was used. By induction hypothesis sequence $\vdash A$ is provable in $\text{GS-}S5_n(\text{ED})$. Then conclusion $K_i A$ of the rule R2 is provable in such a way:

$$\frac{\vdash A}{\vdash K_i A} (\vdash K_i)$$

- R1 and R3 rules are considered similarly.

“ \Leftarrow ”. Analogously as in [7] for logic $S4_n(\text{D})$.

4 Kanger style sequent calculus for logic $S5_n(\text{ED})$

Stig Kanger proposed to use indexes to get cut free sequent calculus for $S5$ [3]. Kanger style index is defined as natural number. In [6] Aida Pliuškevičienė suggested to use ordered pair $\langle i, l \rangle$, where $i \in \{1, 2, \dots, n\}$ and l is either zero or an arbitrary natural number, to get cut free sequent calculus for $S5_2$.

Let p be a propositional symbol, then p^{Index} is an indexed propositional symbol. If $Index$ is any index, then indexation procedure of formula in $S5_n(\text{ED})$ is as follows:

- $(p^{\langle i, r \rangle})^{Index} = p^{\langle i, r \rangle, Index}$
- $(A \wedge B)^{Index} = A^{Index} \wedge B^{Index}$, $(A \vee B)^{Index} = A^{Index} \vee B^{Index}$, $(A \rightarrow B)^{Index} = A^{Index} \rightarrow B^{Index}$, $(\neg A)^{Index} = \neg(A)^{Index}$
- $(K_i A)^{Index} = K_i(A)^{Index}$, $(EA)^{Index} = E(A)^{Index}$, $(DA)^{Index} = D(A)^{Index}$

It is assumed that $p^{Index_1, \langle i, 0 \rangle, Index_2} = p^{Index_1, Index_2}$ and $p^{Index_1, \langle i, l \rangle, \langle i, r \rangle, Index_2} = p^{Index_1, \langle i, r \rangle, Index_2}$.

Kanger style calculus $\text{KS-}S5_n(\text{ED})$ for logic $S5_n(\text{ED})$ is as follows:

- Axiom:
 - $\Gamma, A \vdash \Delta, A$,
 - $\Gamma, A^{Index} \vdash \Delta, A^{Index}$,
 - $\Gamma, A^{\langle i, r \rangle} \vdash \Delta, A^{\langle G, r \rangle}$;
- Inference rules of the sequent calculus G for propositional logic [1];

- Inference rules for operators K_i , E, D:

$$\frac{\Gamma, A^{(i,\alpha)}, K_i A \vdash \Delta}{\Gamma, K_i A \vdash \Delta} (K_i \vdash), \quad \frac{\Gamma \vdash \Delta, A^{(i,r)}}{\Gamma \vdash \Delta, K_i A} (\vdash K_i),$$

$$\frac{\Gamma, K_1 A \wedge \dots \wedge K_n A \vdash \Delta}{\Gamma, EA \vdash \Delta} (E \vdash), \quad \frac{\Gamma \vdash K_1 A \wedge \dots \wedge K_n A, \Delta}{\Gamma \vdash EA, \Delta} (\vdash E),$$

$$\frac{\Gamma, A^{(G,\alpha)}, DA \vdash \Delta}{\Gamma, DA \vdash \Delta} (D \vdash), \quad \frac{\Gamma \vdash \Delta, A^{(G,r)}}{\Gamma \vdash \Delta, DA} (\vdash D),$$

$$\frac{\Gamma_2 \vdash A^{(G,r)}}{\Gamma_1, K\Gamma_2 \vdash \Delta, DA} (\text{Interaction}).$$

A natural number $r \in \mathbb{N}$ is such that any index in the conclusion does not contain a pair $\langle i, r \rangle$. A value of the metavariable α is either zero or natural number defined as follows: if any index in the conclusion does not contain a pair of the shape $\langle i, b \rangle$ where b is an arbitrary natural number, then α is zero, otherwise value of metavariable α is natural number $l \in \mathbb{N}$ such that $\langle i, l \rangle$ enters in some index of the conclusion. Index G means all indexes of agents in a group. We consider only one group of agents, so there is only one special index G .

As in [6] the following theorem can be proved.

Theorem 1. *The cut rule is admissible in KS-S5_n(ED).*

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REZIU M Ė

Sekvencinis skaičiavimas be pjūvio taisyklės logikai $S5_n(ED)$

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Darbe nagrinėjami Hilbert'o tipo, Gentzen'o tipo sekvencinis ir Kanger'io tipo sekvencinis skaičiavimai žinių logikai $S5_n(ED)$. Pateikiamas Gentzen'o tipo sekvencinis skaičiavimas ir įrodomas jo ekvivalentumas su Hilbert'o tipo skaičiavimu, gaunant Gentzen'o tipo skaičiavimo pagrįstumą ir pilnumą. Suformuojamas indeksinis Kanger'io tipo sekvencinis skaičiavimas be pjūvio taisyklės.

Raktiniai žodžiai: žinių logika, paskirstytas žinojimas, visuotinis žinojimas, sekvencinis skaičiavimas, pjūvio pašalinamumas.