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Value-distribution of twisted L-functions of normalized cusp forms

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Abstract. A limit theorem in the sense of weak convergence of probability measures on the complex plane for twisted with Dirichlet character L-functions of holomorphic normalized Hecke eigen cusp forms with an increasing modulus of the character is proved.

Keywords: Dirichlet character; Hecke eigen form; twisted L-functions.

1 Introduction

Let $q \in \mathbb{N}$, and let $\chi(m)$ denote a Dirichlet character modulo q. Then the twisted L-function $L(s, F, \chi)$ attached to the holomorphic normalized Hecke eigen cusp form $F(z)$ of weight κ for the full modular group is defined, in the half-plane $\sigma > \frac{\kappa+1}{2}$, by the Dirichlet series

$$L(s, F, \chi) = \sum_{m=1}^{\infty} \frac{c(m)\chi(m)}{m^s}, \quad s = \sigma + it.$$

Here

$$F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}, \quad c(1) = 1,$$

is the Fourier series expansion for $F(z)$. The function $L(s, F, \chi)$ can be analytically continued to an entire function. Also, in the half-plane $\sigma > \frac{\kappa+1}{2}$, it can be represented by the Euler product

$$L(s, F, \chi) = \prod_p \left(1 - \frac{\alpha(p)\chi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)\chi(p)}{p^s}\right)^{-1} \tag{1}$$

over primes p. The complex numbers $\alpha(p)$ and $\beta(p)$ satisfy $\alpha(p)\beta(p) = 1$, $\beta(p) = \overline{\alpha(p)}$, and $\alpha(p) + \beta(p) = c(p)$.

For $Q \geqslant 2$, define

$$M_Q = \sum_{q \leqslant Q} \sum_{\substack{\chi = \chi (\mathrm{mod}\, q) \\ \chi \neq \chi_0}} 1,$$

where χ_0 denotes the principal character mod q. For brevity, let

$$\mu_Q(\dots) = M_Q^{-1} \sum_{q\leqslant Q} \sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0 \\ \dots}} 1,$$

where in place of dots a condition satisfied by a pair $(q, \chi(\mathrm{mod}\, q))$ is to be written.

The aim of this note is a generalization to the space $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ of limit theorems with an increasing prime modulus q for $|L(s, F, \chi)|$ and $\arg L(s, F, \chi)$ (see, [3] and [4], respectively). We recall that the function

$$w(\tau, k) = \int_{\mathbb{C}\setminus\{0\}} |z|^{i\tau} e^{ik \arg z}\, dP, \quad \tau \in \mathbb{R},\ k \in \mathbb{Z},$$

is a characteristic transform of the probability measure P on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ and the measure P is uniquely determined by its characteristic transform $w(\tau, k)$.

Let P and P_n, $n \in \mathbb{N}$, be a probability measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$. We say that P_n converges weakly in sense of $\mathbb{C}$ to P if P_n converges weakly to P as $n \to \infty$, and, additionally,$\lim_{n\to\infty} P_n(\{0\}) = P(\{0\})$.

For $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$, let

$$\xi = \xi(\tau, \pm k) = \frac{i\tau \pm k}{2},$$

and, for primes p and $l \in \mathbb{N}$,

$$d_{\tau,\pm k}(p^l) = \frac{\xi(\xi+1)\dots(\xi+l-1)}{l!}, \quad d_{\tau,k}(1) = 1.$$

Define

$$a_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,k}(p^j)\, \alpha^j(p)\, d_{\tau,k}(p^{l-j})\, \beta^{l-j}(p),$$

$$b_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,-k}(p^j)\, \overline{\alpha}^j(p)\, d_{\tau,-k}(p^{l-j})\, \overline{\beta}^{l-j}(p),$$

and for $m \in \mathbb{N}$, let

$$a_{\tau,k}(m) = \prod_{p^l \| m} a_{\tau,k}(p^l), \qquad b_{\tau,k}(m) = \prod_{p^l \| m} b_{\tau,k}(p^l).$$

Thus $a_{\tau,k}(m)$ and $b_{\tau,k}(m)$ are multiplicative arithmetical functions.

Let $P_{\mathbb{C}}$ be a probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ defined by the characteristic transform

$$w(\tau, k) = \sum_{m=1}^{\infty} \frac{a_{\tau,k}(m) b_{\tau,k}(m)}{m^{2\sigma}}, \quad \sigma > \frac{\kappa+1}{2},$$

and let the modulus q of χ be prime.

Define

$$P_{Q,\mathbb{C}}(A) = \mu_Q\big(L(s, F, \chi) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem 1. *Let $\sigma > \frac{\kappa+1}{2}$. Then the probability measure $P_{Q,\mathbb{C}}$ converges weakly in sense of $\mathbb{C}$ to the measure $P_{\mathbb{C}}$ as $Q \to \infty$.*

2 Proof of Theorem 1

We give a shortened proof of Theorem 1. At first, we define the characteristic transformation $w_Q(\tau,k)$ of the probability measure $P_{Q,\mathbb{C}}$, and later we give its asymptotic formula. The definition of $P_{Q,\mathbb{C}}$ implies that, for $\tau\in\mathbb{R}$ and $k\in\mathbb{Z}$,

$$w_Q(\tau,k)=\frac{1}{M_Q}\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}}\bigl|L(s,F,\chi)\bigr|^{i\tau}e^{ik\arg L(s,F,\chi)}. \tag{2}$$

Note that, in view of the Euler product (1) for $L(s,F,\chi)$ and Deligne's estimates

$$\bigl|\alpha(p)\bigr|\leqslant p^{\frac{\kappa-1}{2}},\qquad \bigl|\beta(p)\bigr|\leqslant p^{\frac{\kappa-1}{2}}, \tag{3}$$

$L(s,F,\chi)\neq 0$ for $\sigma>\frac{\kappa+1}{2}$. For $\delta>0$, let $R=\{s\in\mathbb{C}\colon \sigma\geqslant\frac{\kappa+1}{2}+\delta\}$. Since

$$\bigl|L(s,F,\chi)\bigr|=\bigl(L(s,F,\chi)\overline{L(s,F,\chi)}\,\bigr)^{\frac12}\quad\text{and}\quad e^{i\arg L(s,F,\chi)}=\left(\frac{L(s,F,\chi)}{\overline{L(s,F,\chi)}}\right)^{\frac12},$$

from (1) we have that, for $s\in R$,

$$\begin{aligned}
&\bigl|L(s,F,\chi)\bigr|^{i\tau}e^{ik\arg L(s,F,\chi)}\\
&\quad=\prod_p\left(1-\frac{\alpha(p)\chi(p)}{p^s}\right)^{-\frac{i\tau+k}{2}}\left(1-\frac{\beta(p)\chi(p)}{p^s}\right)^{-\frac{i\tau+k}{2}}\\
&\quad\times\prod_p\left(1-\frac{\overline{\alpha}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{-\frac{i\tau-k}{2}}\left(1-\frac{\overline{\beta}(p)\overline{\chi}(p)}{p^{\overline{s}}}\right)^{-\frac{i\tau-k}{2}}.
\end{aligned} \tag{4}$$

Here the multi-valued functions $\log(1-z)$ and $(1-z)^{-w}$, $w\in\mathbb{C}\backslash\{0\}$, in the region $|z|<1$ are defined by continuous variation along any path in this region from the values $\log(1-z)|_{z=0}=0$ and $(1-z)^{-w}|_{z=0}=1$, respectively.

Using the above notation, we have that, for $|z|<1$,

$$(1-z)^{-\xi}=\sum_{l=0}^{\infty}d_{\tau,\pm k}\bigl(p^l\bigr)z^l.$$

Therefore, (4) shows that, for $s\in R$,

$$\begin{aligned}
\bigl|L(s,F,\chi)\bigr|^{i\tau}e^{ik\arg L(s,F,\chi)}&=\prod_p\sum_{j=0}^{\infty}\frac{d_{\tau,k}(p^j)\alpha^j(p)\chi(p^j)}{p^{js}}\sum_{l=0}^{\infty}\frac{d_{\tau,k}(p^l)\beta^l(p)\chi(p^l)}{p^{ls}}\\
&\quad\times\prod_p\sum_{j=0}^{\infty}\frac{d_{\tau,-k}(p^j)\overline{\alpha}^j(p)\overline{\chi}(p^j)}{p^{j\overline{s}}}\sum_{l=0}^{\infty}\frac{d_{\tau,-k}(p^l)\overline{\beta}^l(p)\overline{\chi}(p^l)}{p^{l\overline{s}}}\\
&=\sum_{m=1}^{\infty}\frac{\hat{a}_{\tau,k}(m)}{m^s}\sum_{n=1}^{\infty}\frac{\hat{b}_{\tau,k}(n)}{n^{\overline{s}}},
\end{aligned} \tag{5}$$

where $\hat{a}_{\tau,k}(m)$ and $\hat{b}_{\tau,k}(m)$ are multiplicative functions defined, for primes p and $l \in \mathbb{N}$, by

$$\hat{a}_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,k}(p^j)\alpha^j(p)\chi(p^j)d_{\tau,k}(p^{l-j})\beta^{l-j}(p)\chi(p^{l-j}) \tag{6}$$

and

$$\hat{b}_{\tau,k}(p^l) = \sum_{j=0}^{l} d_{\tau,-k}(p^j)\overline{\alpha}^j(p)\overline{\chi}(p^j)d_{\tau,-k}(p^{l-j})\overline{\beta}^{l-j}(p)\overline{\chi}(p^{l-j}). \tag{7}$$

For $|\tau| \leqslant c$ and $l \in \mathbb{N}$,

$$\left|d_{\tau,\pm k}(p^l)\right| \leqslant \frac{|\xi|(|\xi|+1)\dots(|\xi|+l-1)}{l!} \leqslant \exp\left\{|\xi|\sum_{v=1}^{l}\frac{1}{v}\right\} \leqslant (l+1)^{c_1}$$

with a suitable positive constant c_1 depending on c and k, only. This, estimates (3), and (6)–(7) imply, for $|\tau| \leqslant c$ and $l \in \mathbb{N}$, the bounds

$$\left|\hat{a}_{\tau,k}(p^l)\right| \leqslant (l+1)^{c_2} p^{\frac{l(\kappa-1)}{2}} \quad \text{and} \quad \left|\hat{b}_{\tau,k}(p^l)\right| \leqslant (l+1)^{c_2} p^{\frac{l(\kappa-1)}{2}}$$

with a positive constant c_2 depending on c and k. Therefore, by the multiplicativity of $\hat{a}_{\tau,k}(m)$ and $\hat{b}_{\tau,k}(m)$, for $m \in \mathbb{N}$,

$$\left|\hat{a}_{\tau,k}(m)\right| = \prod_{p^l \| m} \left|\hat{a}_{\tau,k}(p^l)\right| \leqslant m^{\frac{\kappa-1}{2}} d^{c_2}(m), \tag{8}$$

and

$$\left|\hat{b}_{\tau,k}(m)\right| = \prod_{p^l \| m} \left|\hat{b}_{\tau,k}(p^l)\right| \leqslant m^{\frac{\kappa-1}{2}} d^{c_2}(m), \tag{9}$$

where $d(m)$ is the classical divisor function.

Now we give an asymptotic formula for the characteristic transform $w_Q(\tau,k)$ as $Q \to \infty$. Let $r = \log Q$. It is well known that $d(m) = O_\varepsilon(m^\varepsilon)$ with every positive ε. Therefore, for $s \in R$, $|\tau| \leqslant c$ and any fixed $k \in \mathbb{Z}$, estimates (8) and (9) yield

$$\sum_{m>r} \frac{\hat{a}_{\tau,k}(m)}{m^s} = O_\varepsilon(r^{-\delta+\varepsilon}) \quad \text{and} \quad \sum_{m>r} \frac{\hat{b}_{\tau,k}(m)}{m^s} = O_\varepsilon(r^{-\delta+\varepsilon}).$$

Substituting this in (5), we find that

$$\begin{aligned}&\left|L(s,F,\chi)\right|^{i\tau} e^{ik\arg L(s,F,\chi)}\\&\quad= \left(\sum_{m<r} \frac{\hat{a}_{\tau,k}(m)}{m^s} + O_\varepsilon(r^{-\delta+\varepsilon})\right)\left(\sum_{m<r} \frac{\hat{b}_{\tau,k}(m)}{m^s} + O_\varepsilon(r^{-\delta+\varepsilon})\right).\end{aligned}$$

Thus, in view of (2), for $s \in R$, $|\tau| \leqslant c$ and any fixed $k \in \mathbb{Z}$,

$$w_Q(\tau,k) = \frac{1}{M_Q}\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} \left(\sum_{m\leqslant r}\frac{\hat{a}_{\tau,k}(m)}{m^s}\sum_{n\leqslant r}\frac{\hat{b}_{\tau,k}(n)}{n^{\overline{s}}}\right) + O_\varepsilon(r^{-\delta+\varepsilon}), \tag{10}$$

since the estimates

$$\sum_{m\leqslant r}\frac{\hat{a}_{\tau,k}(m)}{m^s}=O(1)\quad\text{and}\quad\sum_{m\leqslant r}\frac{\hat{b}_{\tau,k}(m)}{m^s}=O(1)$$

hold. However, (6)–(7) and the definitions of $a_{\tau,k}(m)$ and $b_{\tau,k}(m)$, show that

$$\hat{a}_{\tau,k}(m)=\prod_{p^l\|m}\chi^l(p)\sum_{j=0}^{l}d_{\tau,k}(p^j)\alpha(p^j)d_{\tau,k}(p^{l-j})\beta(p^{l-j})=a_{\tau,k}(m)\chi(m)$$

and

$$\hat{b}_{\tau,k}(m)=b_{\tau,k}(m)\overline{\chi}(m).$$

Therefore, by (10), for $s\in R$, $|\tau|\leqslant c$ and any fixed $k\in\mathbb{Z}$,

$$w_Q(\tau,k)=\frac{1}{M_Q}\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\,q)\\ \chi\neq\chi_0}}\chi(m)\overline{\chi}(n)\biggl(\sum_{m\leqslant r}\frac{a_{\tau,k}(m)}{m^s}\sum_{n\leqslant r}\frac{b_{\tau,k}(n)}{n^{\overline{s}}}\biggr)+O_\varepsilon\bigl(r^{-\delta+\varepsilon}\bigr). \tag{11}$$

It is easily seen that, for $m=n$, $m\leqslant r$, as $Q\to\infty$,

$$\frac{1}{M_Q}\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\,q)\\ \chi\neq\chi_0}}\chi(m)\overline{\chi}(n)=1-\frac{1}{M_Q}\sum_{\substack{q|m\\ q\leqslant r}}(q-2)=1+o(1), \tag{12}$$

since [2]

$$M_Q=\frac{Q^2}{2\log Q}+O\biggl(\frac{Q^2}{\log^2 Q}\biggr).$$

If $(m,q)=1$, then

$$\sum_{\chi=\chi(\mathrm{mod}\,q)}\chi(m)\overline{\chi}(n)=\begin{cases}q-1 & \text{if } m\equiv n(\mathrm{mod}\,q),\\ 0 & \text{if } m\not\equiv n(\mathrm{mod}\,q).\end{cases}$$

Therefore, for $m\neq n$, $m,n\leqslant r$,

$$\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\,q)\\ \chi\neq\chi_0}}\chi(m)\overline{\chi}(n)=\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\,q)\\ q|(m-n)}}\chi(m)\overline{\chi}(n)+\sum_{q\leqslant Q}\sum_{\substack{\chi=\chi(\mathrm{mod}\,q)\\ q\nmid(m-n)}}\chi(m)\overline{\chi}(n)$$
$$+O\biggl(\frac{Q}{\log Q}\biggr)+O\biggl(\sum_{q\leqslant r}q\biggr)=O\biggl(\frac{Q}{\log Q}\biggr).$$

This together with (11) and (12) shows that, for $s\in R$, $|\tau|\leqslant c$ and any fixed $k\in\mathbb{Z}$,

$$w_Q(\tau,k)=\sum_{m=1}^{\infty}\frac{a_{\tau,k}(m)b_{\tau,k}(m)}{m^{2\sigma}}+o(1), \tag{13}$$

as $Q\to\infty$.

The assertion of Theorem 1 follows from (13) and well-known continuity theorem for characteristic transforms of probability measures on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ [1].

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REZIUMĖ

Normuotų parabolinių formų L-funkcijų sąsūkų reikšmių pasiskirstymas

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Straipsnyje įrodyta ribinė teorema tikimybinių matų silpno konvergavimo prasme normuotų parabolinių formų L-funkcijų sąsūkoms kompleksinėje plokštumoje.

Raktiniai žodžiai: Dirichlė charakteris; Hekės tikrinė forma; L-funkcijų sąsūka.