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Contraction-free calculi for modal logics S5 and KD45

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Abstract. It is known that termination and backtracking are among the most important problems in constructing derivations in non-classical logics. In this paper contraction-free and backtracking-free sequent calculi for modal logics S5 and KD45 are presented and founded. These are index-style sequent calculi with some specific indexed axioms. The main new ideas in considered calculi are to use metavariables (along with natural numbers) as elements of indexes and to numerate by different natural number all the positive occurrences of modality $\Box$.

Keywords: modal logic, sequent calculus, termination, backtracking, contraction-free.

1 Introduction

A proof that suitable logical calculus allows us to get a decision procedure is crucial but it is not enough. Check of termination of a derivation as well as backtracking are very important problems. In [4] termination is ensured using the notion of history. Contraction-free calculi were constructed for intuitionistic logic in [3, 5] and for modal logic S4 in [6], however only for sequents in Mints normal form. In [9] a contraction-free and backtracking-free calculus for S5 is presented, however the reflexivity rule of the calculus includes repetition of the main formula and the second part of adequateness theorem is not proven.

In this paper the transitive logics S5 and KD45 are considered. These (and their multimodal counterparts) are extremely important in various applications in computer science and artificial intelligence. S5 is used in reasoning about knowledge and KD45 is the base for BDI logics.

The aim of this paper is to construct contraction-free and backtracking-free sequent calculi for considered modal logics. To construct such calculi some index-style invertible calculi with some specific axioms are constructed. The main new points in considered index-style modal calculi are (1) using metavariables (along with natural numbers) as elements of indexes and (2) numerating all the positive occurrences of modality $\Box$ by different natural numbers. Metavariables and natural numbers as indexes are used in a similar way in free-variable tableaux calculi in [2]. The modal rules of constructed calculi are contraction free (they do not have duplication of the main formula in the premises), moreover new modal rules are invertible.

2 Initial Hilbert-style and Gentzen-style calculi

2.1 S5 Case

The definition of sequent calculus for S5 used here is based on [8], where the cut rule is used.

Definition 1. Gentzen-style calculus GS5 for S5 is obtained from traditional invertible Gentzen-style calculus for classical propositional logic by adding the following rules for modality $\Box$:

$$\frac{A, \Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to) \qquad \frac{\Box \Gamma \to \Box \Delta, A}{\Pi_1, \Box \Gamma \to \Box \Delta, \Box A, \Pi_2}\ (\to \Box)$$

$$\frac{\Gamma \to \Delta, \Box A \qquad \Box A, \Pi \to \Theta}{\Gamma, \Pi \to \Delta, \Theta}\ (\text{Cut}_\Box)$$

It is obvious that the rule $(\text{Cut}_\Box)$ is necessary for derivability of some sequents, for example, $P \to \Box\neg\Box\neg P$. The necessity to duplicate the main formula in application of rule $(\Box \to)$ can be demonstrated by sequent $\Box\neg(P \vee \Box\neg P) \to$, which is valid in S5.

2.2 KD45 Case

Definition 2. Gentzen-style calculus GKD45 for KD45 is obtained from traditional invertible Gentzen-style calculus for classical propositional logic by adding the single rule:

$$\frac{\Box \Gamma_1, \Gamma_2 \to \Box \Delta, \Theta, \Box \Theta}{\Pi_1, \Box \Gamma_1, \Box \Gamma_2 \to \Box \Delta, \Box \Theta, \Pi_2}\ (\Box)$$

where Θ contains at most one formula.

The repetition of the main formula in the premise of the rule $(\Box)$ is needed as can be demonstrated by sequent $\to \Box(\Box P \supset P)$, which is valid in KD45.

Calculus GKD45 is based on the one provided in [10].

It is well known that both considered Gentzen-style calculi are sound and complete. Moreover in both considered Gentzen-style calculi the (Cut) rule is admissible. It is easy to verify that both considered calculi contain non-invertible rule.

3 Contraction-free and backtracking-free calculi

In all the calculi considered below along with the set of non-indexed propositional symbols P, Q, P_1, $Q_1, \ldots$, the set of indexed propositional symbols P^γ, Q^{γ_1}, P_1^γ, $Q_1^{\gamma_1}, \ldots$ are used, where the index γ_i is a natural number, a metavariable or empty (denoted $\gamma_1 = \varnothing$).[1] Two sorts of metavariables are introduced: simple ones $(\alpha, \alpha_1, \ldots)$ and branching ones $(\beta, \beta_1, \ldots)$. In [7] dummy variables are introduced, which are analogous to metavariables.

[1] Empty index means that no index is used. That is, for some propositional variable p^γ if $\gamma = \varnothing$, then $p^\gamma = p$.

In this article notation A^γ is used, where A is some formula, therefore it is important to define operations with indexes: (1) $(p^{\gamma_1})^{\gamma_2} = p^{\gamma_2}$. (2) $(A \odot B)^\gamma = A^\gamma \odot B^\gamma$, $\odot \in \{\wedge, \vee, \supset\}$. (3) $(\triangle A)^\gamma = \triangle(A^\gamma)$, $\triangle \in \{\neg, \Box\}$.

$\Box A$, which can be part of succedent of any sequent of any possible derivation, must be numbered. However in order to make the numbering differ from formula indexes, all the positive occurrences of modality $\Box$ are numbered instead. It should be stressed that in the presented calculi the numeration of positive occurrences of the modality $\Box$ allows us to numerate in a new way succedental modal rule.

Definition 3. The contraction-free and backtracking-free calculus $\mathrm{CF_{S5}}$ for modal logic S5 consists of traditional invertible propositional rules with single premise and the rules:

$$\frac{\Gamma \to \Delta, A^{\gamma'} \qquad \Gamma \to \Delta, B^{\gamma'}}{\Gamma \to \Delta, (A \wedge B)^\gamma}\ (\to \wedge) \qquad \frac{A^{\gamma'}, \Gamma \to \Delta \qquad B^{\gamma'}, \Gamma \to \Delta}{(A \vee B)^\gamma, \Gamma \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to \Delta, A^{\gamma'} \qquad B^{\gamma'}, \Gamma \to \Delta}{(A \supset B)^\gamma, \Gamma \to \Delta}\ (\supset \to)$$

$$\frac{A^\alpha, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box^\alpha \to) \qquad \frac{\Gamma \to \Delta, A^l}{\Gamma \to \Delta, \Box^l A}\ (\to \Box^l)$$

Here γ is any index, if γ is simple metavariable, then γ' is new branching metavariable (the one, that is not part of the derivation yet), otherwise $\gamma' = \gamma$ and α is a simple metavariable.[2] The rule $(\Box^\alpha \to)$ is called α-rule, and the rule $(\to \Box^l)$ is called l-rule.

An axiom of the calculus is sequent $\Gamma, p^{\gamma_1} \to p^{\gamma_2}, \Delta$ and if one of the indexes γ_1 and γ_2 is natural number then either the other one is natural number and $\gamma_1 = \gamma_2$, or the other one is a metavariable.[3] Propositional symbols p^{γ_1} and p^{γ_2} are main formulas of an axiom.

Furthermore, the traditional concept of sucesfull derivation is further limited by Condition $(*)$. As usual, if at least one branch of the derivation is not an axiom and no rule of the calculus can be applied, then the derivation is not succesful. However, if every branch ends with an axiom, then the compatibility of those axioms must be checked. First of all, a pair of main formulas must be selected in every axiom. If p^{β_1} and p^{β_2} are the main formulas of some axiom, then it is said that β_1 and β_2 are equivalent and denoted $\beta_1 \sim \beta_2$. The relation $\sim$ is reflexive, symetric and transitive.

Now, let S_1 and S_2 be two axioms of the same derivation. Additional requirements for compatibility apply only, if in both axioms one of the two main formulas is indexed by branching metavariable, another one by natural number and the branching metavariables of those axioms are equivalent. Otherwise, S_1 and S_2 are compatible

[2] It is not necessary to have different instances of simple metavariable, therefore only one simple metavariable is needed. However with each application of branching rule ($(\to \wedge)$, $(\vee \to)$ or $(\supset\to)$) a new branching metavariable must be used.

[3] The possible variants of γ_1 and γ_2 are: (1) They both are metavariables. (2) One is a metavariable, another one is natural number or empty. (3) They both are natural numbers and $\gamma_1 = \gamma_2$. (4) They both are empty.

without further check. Now let p^{β_1} and p^{l_1} be the main formulas of S_1, q^{β_2} and q^{l_2} be the main formulas of S_2 and $\beta_1 \sim \beta_2$. Then S_1 and S_2 are compatible, if:

- $l_1 = l_2$ or
- if p^{β_1} is part of antecedent of S_1, then it is also part of antecedent of S_2. Otherwise it is part of succedent of S_2. And if q^{β_2} is part of antecedent of S_2, then it is part of antecedent of S_1. Otherwise, it is part of succedent of S_1.

The Condition $(*)$ simply states that any two axioms must be compatible.

Propositional rules $(\to \wedge)$, $(\vee \to)$ and $(\supset\to)$ have two presumptions and therefore are called the branching rules. It should be noted, that their only difference from the traditional ones is that if simple metavariable is part of the main formula, than it is changed to new branching metavariable in the premises.

Moreover, in the initial sequent of the derivation in $\mathrm{CF_{S5}}$ every positive occurence of $\Box$ must be indexed. This ensures the correct application of the l-rule.

Using traditional proof-theoretical technique it can be proved that all the rules of $\mathrm{CF_{S5}}$ are invertible and that the (Cut) rule is admissible in $\mathrm{CF_{S5}}$. Since modal rules do not contain duplication of the main formula, $\mathrm{CF_{S5}}$ is contraction-free. Moreover, because of ivertibility, the rules of $\mathrm{CF_{S5}}$ can be applied in any order and no backtracking is needed to check whether application of different rule to the same sequent would yield different result. Therefore, $\mathrm{CF_{S5}}$ is backtracking-free, however it should be stressed that some elements of backtracking are involved in axiom searches.

Definition 4. The contraction-free and backtracking-free calculus $\mathrm{CF_{KD45}}$ for modal logic KD45 is the same as calculus $\mathrm{CF_{S5}}$, except that one more restriction is added to the axiom. An axiom of the calculus is sequent $\Gamma, p^{\gamma_1} \to p^{\gamma_2}, \Delta$ and:

1. If one of indexes γ_1 and γ_2 is natural number then either the other one is number and $\gamma_1 = \gamma_2$, or the other one is a metavariable.
2. If one of indexes γ_1 and γ_2 is metavariable then the other one is either natural number or metavariable.

It should be noticed, that the only difference between $\mathrm{CF_{KD45}}$ and $\mathrm{CF_{S5}}$ definitions is that sequents $\Gamma, p^{\alpha} \to p, \Delta$ and $\Gamma, p \to p^{\alpha}, \Delta$ (in this case α is either simple or branching metavariable) are axioms of $\mathrm{CF_{S5}}$, but not axioms of $\mathrm{CF_{KD45}}$.

4 Foundation of the calculi

Using traditional proof-theoretical technique (see e.g. [1, 10]) we can prove that all structural rules (including (Cut)) are admissible in both $\mathrm{CF_{S5}}$ and $\mathrm{CF_{KD45}}$. Using admissibility of (Cut), the following can be proved.

Lemma 1. *For each sequent S we have that if* $\mathrm{G}\mathfrak{I} \vdash S$ *then* $\mathrm{CF}_{\mathfrak{I}} \vdash S$*, where* $\mathfrak{I} \in \{\mathrm{S5}, \mathrm{KD45}\}$.

To prove inverse statement let us introduce an auxiliary hybrid calculi $\mathrm{CF}'_{\mathfrak{I}}$, where $\mathfrak{I} \in \{\mathrm{S5}, \mathrm{KD45}\}$ obtained from $\mathrm{G}\mathfrak{I}$ by adding the α-rule and branching rules of the calculus $\mathrm{CF}_{\mathfrak{I}}$, correspondingly to each logic $\mathfrak{I}$. However sequents are allowed to contain any indexes. Additional requirements for axiom of $\mathrm{CF}_{\mathfrak{I}}$ and Condition $(*)$ is also part of $\mathrm{CF}'_{\mathfrak{I}}$. It is easy to prove admissibility of (Cut) in $\mathrm{CF}'_{\mathfrak{I}}$.

Let $\Pi_1 = A_1, \ldots, A_n$ and $\Pi_2 = B_1, \ldots, B_m$. Then $\Pi_1 \supset \Pi_2 = \bigwedge_{i=1}^{n} A_i \supset \bigvee_{j=1}^{m} B_j$, if $n = 0$, then $\Pi_1 \supset \Pi_2 = \bigvee_{j=1}^{m} B_j$, and if $m = 0$, then $\Pi_1 \supset \Pi_2 = \neg \bigwedge_{i=1}^{n} A_i$.

Let $\mathrm{CF}'_{\mathfrak{I}} \vdash S$ and let the derivation of S be V. Let formula A be a member of S. We say that A is essential in V if at least one propositional symbol from A is main formula of an axiom of V.

Lemma 2 [Replacing indexes by modality]. *Let's say we have a sequent* $S = \Pi_1, \Gamma \rightarrow \Pi_2, \Delta$, *where* $\Gamma = \Gamma_1^{\gamma_{11}}, \ldots, \Gamma_n^{\gamma_{1n}}$, $\Delta = \Delta_1^{\gamma_{21}}, \ldots, \Delta_m^{\gamma_{2m}}$, $\gamma_{ij} \neq \varnothing, \gamma_{ij} \notin \{\Pi_1, \Pi_2\}$. *If* $\mathrm{CF}'_{\mathfrak{I}} \vdash S$, $\mathfrak{I} \in \{\mathrm{S5}, \mathrm{KD45}\}$, *and at least one formula from* Γ, Δ *is essential, then* $\mathrm{CF}'_{\mathfrak{I}} \vdash S^* = \ \rightarrow \Box^{\circ}(\Pi_1 \supset \Pi_2), \Box(\Gamma \supset \Delta)$, *where* $\Box^{\circ} = \Box$, *if* $\mathfrak{I} = \mathrm{S5}$, *and* $\Box^{\circ} = \varnothing$, *if* $\mathfrak{I} = \mathrm{KD45}$.

Proof. The proof of the lemma is carried out using induction on the height of the derivation of sequent S in $\mathrm{CF}'_{\mathfrak{I}}$. □

Let $\mathrm{CF}''_{\mathfrak{I}}$, $\mathfrak{I} \in \{\mathrm{S5}, \mathrm{KD45}\}$ be the calculus obtained from $\mathrm{CF}'_{\mathfrak{I}}$ by adding the l-rule.

Lemma 3. *Let* S *be index-free sequent, then if* $\mathrm{CF}''_{\mathfrak{I}} \vdash S$ *then* $\mathrm{G}\mathfrak{I} \vdash S$, *where* $\mathfrak{I} \in \{\mathrm{S5}, \mathrm{KD45}\}$

Proof. The lemma is proved by induction on the number of applications of the rule $(\rightarrow \Box^{l})$ in given the derivation of sequent S, using Lemma 2. □

From Lemmas 1 and 3 we get:

Theorem 1. *The calculi* $\mathrm{CF}_{\mathrm{S5}}$ *and* $\mathrm{CF}_{\mathrm{KD45}}$ *are sound and complete.*

Finally, the following theorem can be easily proved.

Theorem 2. *The calculi* $\mathrm{CF}_{\mathrm{S5}}$ *and* $\mathrm{CF}_{\mathrm{KD45}}$ *are decidable. That is, for any sequent* S *it is possible to say that either* S *is derivable or not derivable after finite number of rule applications.*

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REZIUMĖ

Dubliavimo eliminavimas modalumo logikoms S5 ir KD45

J. Andrikonis, R. Pliuškevičius

Žinoma, kad baigtinumas ir grįžimas išvedimo paieškos medžiu yra kelios iš pačių svarbiausių išvedimų paieškos neklasikinėse logikose problemų. Šitame straipsnyje pateikiami sekvenciniai skaičiavimai modalumo logikoms S5 ir KD45, kuriuose nėra dubliavimo ir išvedimų paieška nereikalauja grįžimo. Tai indeksiniai skaičiavimai su tam tikromis indeksinėmis aksiomomis. Pagrindinės pristatomų skaičiavimų naujovės yra metakintamųjų naudojimas (kartu su natūraliaisiais skaičiais) indeksuose ir visų teigiamų modalumo $\Box$ įeičių numeravimas natūraliais skaičiais.

Raktiniai žodžiai: modalumo logika, sekvencinis skaičiavimas, baigtinumas, grįžimas, be dubliavimo