A sequent calculus for propositional temporal logic with time gaps

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Abstract. A sequent calculus with Kripke semantics internalization for a propositional temporal logic with time gaps is introduced. All rules of the calculus are context-free and height-preserving invertible. Structural rules are admissible. The calculus is cut free and is proved to be complete.

Keywords: temporal logic, sequent calculus, Kripke semantics internalization.

1 Introduction

Standard sequent calculi for modal logics usually have context-dependent, non-invertible derivation rules. Structural rules and cut are often not admissible in such calculi, e.g., in a standard Gentzen-like sequent calculus for the logic $S5$. As a consequence, proof-theory by means of sequent calculi can not be developed.

This problem is solved using various methods, e.g., hypersequents [8], display logic [5]. Excellent results are achieved by Kripke semantics internalization into a sequent calculus method introduced in [6].

In the present paper, we consider Kripke semantics internalization into a sequent calculus of a propositional temporal logic with time gaps introduced in [2]. A new feature of the temporal logic comparing to the modal logics considered in [6] is the operator 'next'. All rules of the introduced sequent calculus $\mathbf{LB}'_0$ are context-free and height-preserving invertible. Structural rules are admissible. The calculus is cut free and is proved to be complete.

Works [3] and [4] investigating linear discrete time logic and covering also the temporal logic with time gaps are closely related to the present work. We use less types of relation atoms and more compact notation of successors of temporal points.

The paper is organized as follows. A Gentzen-like sequent calculus $\mathbf{LB}_0$ for the propositional temporal logic with time gaps is presented in Section 2. The calculus is a slight modification of the propositional variant of $\mathbf{LB}$ in [2]. The sequent calculus $\mathbf{LB}'_0$ is introduced in Section 3. $\mathbf{LB}'_0$ is obtained by Kripke semantics internalization into $\mathbf{LB}_0$. Height-preserving rule invertibility and admissibility of structural rules in $\mathbf{LB}'_0$ is dealt with in Section 4. Completeness of $\mathbf{LB}'_0$ is proved in Section 5.

2 Initial sequent calculus $\mathbf{LB}_0$

The sequent calculus LK_0 is a variant of the propositional Gentzen-like sequent calculus, see, e.g., [1].

The sequent calculus $\mathbf{LB}_0$ of the propositional temporal logic with time gaps [2] is obtained from LK_0 by adding the rules for the temporal operators 'next' and 'always' and the structural rule of weakening:

$$\frac{A, \circ\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta} \ \Box : \text{left}, \qquad \frac{\Gamma \to A, \Delta; \Gamma \to \circ\Box A, \Delta}{\Gamma \to \Box A, \Delta} \ \Box : \text{right},$$

$$\frac{\Gamma \to \Delta}{\circ\Gamma \to \circ\Delta} \ \text{nex}, \qquad \frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A} \ \text{nec},$$

$$\frac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Lambda} \text{W}.$$

Γ, Δ, Π, and Λ denote finite, possibly empty, multisets of formulas.

$\mathbf{LB}_0$ is complete for a certain semantics with branching time gaps [2].

3 Kripke semantics internalization into $\mathbf{LB}_0$

Kripke semantics of $\mathbf{TB}$ temporal logic with branching time gaps is defined as follows.

Let T be an enumerable partially ordered set. T belongs to the class $\mathcal{L}$ of linear discrete orders iff it is order isomorphic to ω; it belongs to the class $\mathcal{B}$ of linear discrete orders with branching gaps iff it is order isomorphic to a well-founded tree of ω-segments.

Temporal points of $T \in \mathcal{B}$ are denoted by u, x, y, z, and w. Next to x point is denoted by x'. x^n stands for $\overbrace{x' \cdots '}^{n}$; $x^0 = x$.

1. $x \models A$ is defined in a common for modal logics way when A is an atomic formula or a formula of the shape $\neg B$ or $B\theta C$, where θ is a logical connective, see [2].

2. $x \models \circ A$ iff $x' \models A$.

3. $x \models \Box A$ iff for all y, $x \leqslant y$ implies $y \models A$.

The sequent calculus $\mathbf{LB}_0'$ is obtained by $\mathbf{TB}$ semantics internalization into $\mathbf{LB}_0$:

1. Axioms: $\Gamma, x^k : E \to x^k : E, \Delta$.

2. Logical rules:

$$\frac{\Gamma \to x^k : A, \Delta; \ \Gamma \to x^k : B, \Delta}{\Gamma \to x^k : A \wedge B, \Delta} \ (\to \wedge), \qquad \frac{x^k : A, x^k : B, \Gamma \to \Delta}{x^k : A \wedge B, \Gamma \to \Delta} \ (\wedge \to),$$

$$\frac{x^k : A, \Gamma \to \Delta; \ \ x^k : B, \Gamma \to \Delta}{x^k : A \vee B, \Gamma \to \Delta} \ (\vee \to), \qquad \frac{\Gamma \to x^k : A, x^k : B, \Delta}{\Gamma \to x^k : A \vee B, \Delta} \ (\to \vee),$$

$$\frac{\Gamma \to x^k : A, \Delta}{x^k : \neg A, \Gamma \to \Delta} \ (\neg \to), \qquad \frac{\Gamma, x^k : A \to \Delta}{\Gamma \to x^k : \neg A, \Delta} \ (\to \neg),$$

$$\frac{\Gamma \to x^k : A, \Delta; \ \ x^k : B, \Gamma \to \Delta}{x^k : A \supset B, \Gamma \to \Delta} \ (\supset \to), \qquad \frac{\Gamma, x^k : A \to x^k : B, \Delta}{\Gamma \to x^k : A \supset B, \Delta} \ (\to \supset).$$

Here E, A and B are the same as in LK_0.

3. Temporal rules:

$$\frac{x^{k+1} : A, \Gamma \to \Delta}{x^k : \circ A, \Gamma \to \Delta} \ (\circ \to), \qquad \frac{\Gamma \to x^{k+1} : A, \Delta}{\Gamma \to x^k : \circ A, \Delta} \ (\to \circ),$$

$$\frac{y^n : A, x^{k+m} \leqslant y^n, x^k : \Box A, \Gamma \to \Delta}{x^{k+m} \leqslant y^n, x^k : \Box A, \Gamma \to \Delta} \ (\Box \to),$$

$$\frac{\Gamma \to x^k : A, \Delta; \ \ x^{k+1} \leqslant y, \Gamma \to y : A, x^{k+1} : \Box A, \Delta}{\Gamma \to x^k : \Box A, \Delta} \ (\to \Box).$$

Here $k, m, n \geqslant 0$; the label '$y^n :$' in $(\Box \to)$ occurs in $x^k : \Box A, \Gamma, \Delta$; y in $(\to \Box)$ does not occur in the conclusion.

4. Rules for order atoms:

$$\frac{x^k \leqslant x^k, \Gamma \to \Delta}{\Gamma \to \Delta} \ \text{Ref}, \qquad \frac{x^k \leqslant y^{m+n}, x^k \leqslant y^m, \Gamma \to \Delta}{x^k \leqslant y^m, \Gamma \to \Delta} \ \text{Fwd},$$

$$\frac{x^l \leqslant z^n, x^l \leqslant y^k, y^{k+m} \leqslant z^n, \Gamma \to \Delta}{x^l \leqslant y^k, y^{k+m} \leqslant z^n, \Gamma \to \Delta} \ \text{Trans}.$$

Here $l, k, m, n \geqslant 0$. In Ref, $x^k \leqslant x^k$ does not occur in Γ. In Fwd, $x^k \leqslant y^{m+n}$ does not occur in Γ. In Trans, $x^l \leqslant z^n$ does not occur in Γ.

An expression of the shape '$x^k : A$' is called a labeled formula. It is understood as $x^k \models A$, i.e., A is true at the time point x^k. x^k in '$x^k : A$' is called a label. An expression $x^k \leqslant y^m$ is called an order atom.

In all the rules of $\mathbf{LB}'_0$: Γ and Δ denote finite, possibly empty, multisets of labeled formulas and order atoms; Δ is order atom free.

Ref, Fwd, and Trans are obtained by transforming corresponding axioms into the rules, using the method provided in [6], see also [7].

A derivation or proof-search tree is denoted by V and its height by $h(V)$. $h(V)$ is equal to the length of the longest branch on V. The length of a branch is measured by the number of rule applications. The top sequent of a derivation tree branch is called a leaf. A tree is called closed iff all its leaves are axioms. $V(S)$ denotes a derivation tree with the end-sequent S.

The following examples show that the sequents $x : \Box A \to x : \circ \circ A$ and $x : \circ \Box A \to x : \Box \circ A$ (A atomic) are derivable in $\mathbf{LB}'_0$.

1.

$$\frac{\dfrac{x^2 : A, x \leqslant x^2, x \leqslant x, x : \Box A \to x^2 : A}{\dfrac{x \leqslant x^2, x \leqslant x, x : \Box A \to x^2 : A}{\dfrac{x \leqslant x, x : \Box A \to x^2 : A}{\dfrac{x : \Box A \to x^2 : A}{x : \Box A \to x : \circ\circ A} \, (\to \circ),\, (\to \circ)} \, \text{Ref}} \, \text{Fwd}} \, (\Box \to)}{}$$

2.

$$\frac{\dfrac{y' : A, x' \leqslant y', x' \leqslant y, x' : \Box A \to y' : A, x' : \Box\circ A}{\dfrac{x' \leqslant y', x' \leqslant y, x' : \Box A \to y' : A, x' : \Box\circ A}{\dfrac{x' \leqslant y, x' : \Box A \to y' : A, x' : \Box\circ A}{\downarrow S = x' \leqslant y, x' : \Box A \to y : \circ A, x' : \Box\circ A} \, (\to \circ)} \, \text{Fwd}} \, (\Box \to)}{}$$

$$\frac{\dfrac{\dfrac{x' : A, x' \leqslant x', x' : \Box A \to x' : A}{\dfrac{x' \leqslant x', x' : \Box A \to x' : A}{\dfrac{x' : \Box A \to x' : A}{x' : \Box A \to x : \circ A} \, (\to \circ)} \, \text{Ref}} \, (\Box \to) \quad\quad S\uparrow}{\dfrac{x' : \Box A \to x : \Box\circ A}{x : \circ\Box A \to x : \Box\circ A} \, (\circ \to)} \, (\to \Box)}{}$$

4   Rule invertibility and admissibility of structural rules

Lemma 1. *Any sequent of the shape* $\Gamma, x^k : B \to x^k : B, \Delta$ *is derivable in* $\mathbf{LB}_0'$*. Here* B *is an arbitrary formula.*

Proof. The lemma is proved by induction on the number of logical and temporal operators in B. Let us consider the inductive case with $B = \Box A$:

$$\frac{\Gamma, y : A, x^{k+1} \leqslant y, x^k : \Box A \to y : A, x' : \Box\circ A, \Delta}{\downarrow S = \Gamma, x^{k+1} \leqslant y, x^k : \Box A \to y : A, x' : \Box\circ A, \Delta} \, (\Box \to)$$

$$\frac{\dfrac{\Gamma, x^k : A, x^k \leqslant x^k, x^k : \Box A \to x^k : A, \Delta}{\dfrac{\Gamma, x^k \leqslant x^k, x^k : \Box A \to x^k : A, \Delta}{\Gamma, x^k : \Box A \to x^k : A, \Delta} \, \text{Ref}} \, (\Box \to) \quad\quad \uparrow S}{\Gamma, x^k : \Box A \to x^k : \Box A, \Delta} \, (\to \Box)$$

Further apply the inductive hypothesis to the leaves of the obtained derivation tree. Cases of another shapes of B are considered similarly. $\square$

Further we will want to substitute a temporal point y^k for some other temporal point x in a sequent. The substitution is defined in the natural way, e.g., $x^k \leqslant y^m(z^n/x) \equiv z^{nk} \leqslant y^m$ and $x^k : A(z^m/y) \equiv x^k : A$ if $x \neq y$.

Lemma 2. $LB_0' \vdash^V \Gamma \to \Delta$ *implies* $LB_0' \vdash^{V'} \Gamma(x^k/y) \to \Delta(x^k/y)$ *and* $h(V') \leqslant h(V)$.

Lemma 3. *The rule of weakening*

$$\frac{\Gamma \to \Delta}{\Pi, \Gamma \to \Delta, \Lambda} W$$

is height-preserving admissible in $\mathbf{LB}_0'$. *Here* Π *and* Λ *are arbitrary multisets.*

Lemma 4. *All the rules of* $\mathbf{LB}_0'$ *are height-preserving invertible.*

Lemma 5. *The rule of contraction*

$$\frac{\theta_1, \theta_1, \Gamma \to \theta_2, \theta_2, \Delta}{\theta_1, \Gamma \to \theta_2, \Delta} Ctr$$

is height-preserving admissible in $\mathbf{LB}_0'$. *Here* θ_i *is a labeled formula, an order atom, or the empty set* (θ_2 *is not an order atom*).

5 Completeness

It is easy to check that all $\mathbf{LB}_0'$ rules are semantically correct. We prove in this section that $\mathbf{LB}_0'$ is complete: $LB_0 \vdash \Gamma \to \Delta$ implies $LB_0' \vdash x : \Gamma \to x : \Delta$.

Lemma 6. $LB_0' \vdash^V S = x : \Box\Pi, \Gamma \to \Delta$ *implies* $LB_0' \vdash x' \leqslant y, x : \Box\Pi, \Gamma(y/x) \to \Delta(y/x)$, *where* $\Box\Pi$ *is any fixed multiset of boxed formulas with the label* x.

Lemma 7. $LB_0' \vdash x : \Box\Gamma \to x : A$ *implies* $LB_0' \vdash x : \Box\Gamma \to x : \Box A$.

Theorem 1 [Completeness]. $LB_0 \vdash^V \Gamma \to \Delta$ *implies* $LB_0' \vdash x : \Gamma \to x : \Delta$.

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REZIUMĖ

Sekvencinis skaičiavimas teiginių laiko logikai su laiko tarpsniais
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Straipsnyje yra pateiktas sekvencinis skaičiavimas teiginių laiko logikai su laiko tarpsniais. Šis skaičiavimas yra gautas Kripkės semantikos internalizacijos būdu. Šiame skaičiavime leistinos struktūrinės bei pjūvio taisyklės. Visos jo taisyklės yra nepriklausomos nuo konteksto ir apverčiamos. Šis skaičiavimas yra korektiškas ir pilnas.

Raktiniai žodžiai: laiko logika, sekvencinis skaičiavimas, Kripkės semantikos internalizacija.