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On positive eigenfunctions of one nonlocal boundary problem

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1. Introduction

In scientific literature much attention is given recently to theoretical analysis of differential problems with various types nonlocal conditions. Quite new area is investigation of the spectrums of differential equations with nonlocal conditions. Eigenvalue problems for differential operators with the nonlocal boundary conditions are considerably less investigated than classical boundary conditions cases. In papers [2,3] the similar problems are investigated for the operators with nonlocal condition of Samarskii–Bitsadze or integral type. There is analysed how the spectrum of these problems depends on nonlocal boundary conditions parameters.

Also in paper [1] the eigenvalue problem is analysed. In this paper Hammerstein integral equation is investigated, and obtained results are applied to second order differential equation with nonlocal boundary conditions of Samarskii–Bitsadze type. Eigenvalues and eigenfunctions of this problem are analysed and theorems of positive eigenvalues and eigenfunctions existence in general case are formulated.

In this paper we present separate cases of Sturm–Liouville problem with nonlocal two-point boundary condition. The existence of positive eigenvalues and eigenfunctions is analysed. Obtained results we compare with results that are described in paper [1] and denote that in some cases they do not coincide.

2. Eigenvalues and eigenfunctions of the Sturm–Liouville problem

Let's analyze the Sturm–Liouville problem with one classical boundary condition and another nonlocal second type boundary condition

$$-u'' = \lambda u, \qquad t \in (0,1), \tag{1}$$

$$u(0) = 0, \quad u(1) = \gamma u'(\xi) \tag{2}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in (0,1)$.

In paper [1] G. Infante studied the existence of eigenvalues of Hammerstein integral equation $\tilde{\lambda} u(t) = \int_G k(t,s) f(s,u(s))\mathrm{d}s$. Where G is a compact set in $\mathbb{R}^n$, k and f are allowed to be discontinuous, and k may change sign. Obtained results are applied to second order differential equation

$$\tilde{\lambda} u''(t) + f(t,u(t)) = 0, \quad (0 < t < 1), \tag{3}$$

with various nonlocal boundary conditions for $0 < \xi < 1$:

$$u'(0) = 0, \quad u(1) = \gamma u'(\xi); \tag{4}$$

$$u(0) = 0, \quad u(1) = \gamma u'(\xi); \tag{5}$$

$$u'(0) = 0, \quad u(1) = \gamma u(\xi); \tag{6}$$

$$u(0) = 0, \quad u(1) = \gamma u(\xi). \tag{7}$$

In paper [1] it is presented that:

- the problem 3–4 has a positive eigenvalue, when $\gamma < 0$ and $0 < \gamma < 1 - \xi$, and a corresponding eigenfunction that is positive on $[0, 1)$, when $\gamma < 0$, and on $[0, \xi]$, when $0 < \gamma < 1 - \xi$;
- the problem 3, 5 has a positive eigenvalue, when $\gamma < 0$ and $0 < \gamma < 1 - \xi$, and a corresponding eigenfunction that is positive on $(0, \xi]$, when $\gamma < 0$, and on $(0, 1 - \gamma]$, when $0 < \gamma < 1 - \xi$;
- the problem 3, 6 has a positive eigenvalue, when $\gamma < 0$ and $0 < \gamma < 1$, and a corresponding eigenfunction that is positive on $[0, \xi]$, when $\gamma < 0$, and $[0, 1]$, when $0 < \gamma < 1$, also this problem has a negative eigenvalue, when $\gamma > 1$ and a corresponding eigenfunction that is negative on $[a, b]$, where $a = \xi$, $b \in (\xi, 1]$;
- the problem 3, 7 has a positive eigenvalue, when $\gamma\xi < 0$ and $0 < \gamma\xi < 1$, and a corresponding eigenfunction that is positive on $(0, \xi]$, when $\gamma\xi < 0$, and $(0, 1]$, when $0 < \gamma\xi < 1$, also this problem has a negative eigenvalue, when $\gamma\xi > 1$ and a corresponding eigenfunction that is negative on $[\xi, 1]$.

In this article problem 1–2 is investigated, which is the separate case of problem 3, 5: $\lambda = \frac{1}{\lambda}$ and $f(\cdot, u(t)) \equiv u(t)$. When $\gamma = 0$ in problem 1–2, we get a problem with classical boundary conditions. Then eigenvalues and eigenfunctions do not depend on the parameter ξ: $\lambda_k = (\pi k)^2$, $u_k(t) = \sin(\pi k t)$, $k \in \mathbb{N}$.

In the general case, for $\lambda \neq 0$, eigenfunctions are $u(t) = c\sin(qt)$ and eigenvalues are $\lambda = q^2$, where $q \in \mathbb{C}_q$, $\mathbb{C}_q := \{q \in \mathbb{C} | \operatorname{Re} q > 0 \text{ or } \operatorname{Re} q = 0, \operatorname{Im} q > 0 \text{ or } q = 0\}$, and q we can find from some characteristic equation or q are some constant values. Further, instead of $q \in \mathbb{C}_q$, q we take only in the ray $q = x \geqslant 0$.

In previous papers [2,3] it is obtained that the eigenvalue $\lambda = 0$ exists if and only if $\gamma = 1$. Also, in problem 1–2 constant eigenvalues (which depend on parameter γ) exist for rational $\xi = \frac{m}{n} \in [0, 1]$, when $m \in \mathbb{N}_o$, $n \in \mathbb{N}_e$, and they are equal to $\lambda_k = c_k^2$, $c_k = \pi(k - \frac{1}{2})n$, $k \in \mathbb{N}$ (where $\mathbb{N}_e = \{k \in \mathbb{N}_2 | k \leqslant n\} \cup \{0\}$ – a set of even nonnegative integer numbers, and $\mathbb{N}_o = \{k \in \mathbb{N} \setminus \mathbb{N}_2 | k \leqslant n\}$ – a set of odd positive integer numbers) [2,3].

All nonconstant eigenvalues (which do not depend on parameter γ) we can obtain from the characteristic equation (see, [2])

$$\gamma(x; \xi) = \frac{\sin x}{x\cos(\xi x)} \quad \text{when } x \geqslant 0, \tag{8}$$

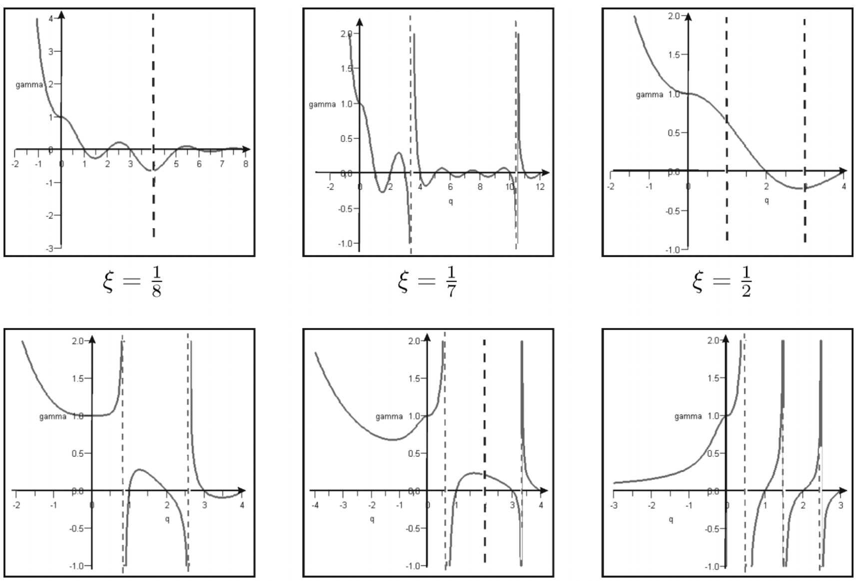
as γ-values of the function γ. Here $\gamma(x;\xi)$ we will denote function, which depends on one variable x, ξ is parameter. The graphs of functions $\gamma(x;\xi)$ for various parameter ξ values are presented in Fig. 1. In the cases $\xi=\frac{1}{8}$, $\xi=\frac{1}{2}$ and $\xi=\frac{3}{4}$ we have constant eigenvalues (dashed vertical lines).

Eigenfunctions are $u(t)=c\sin(xt)$ in the general case for $c>0$, and function $\sin(xt)$ is positive when $xt\in(0,\pi)$. Here x values we can get from the equation $\gamma(x;\xi)=\gamma$. Then in the right side of the interval $t_+x=\pi$, therefore $x=\frac{\pi}{t_+}$ (t_+ we can obtain solving equation $\gamma(\frac{\pi}{t_+})=\gamma$). Further, we will denote $t_+=y$. Consequently positive eigenfunctions existence intervals can be obtained in 8 formula making transformation $x=\frac{\pi}{y}$:

$$\widehat{\gamma}(y;\xi)=\frac{\sin\frac{\pi}{y}}{\frac{\pi}{y}\cos(\xi\frac{\pi}{y})}\quad\text{when } y\geqslant 0. \tag{9}$$

The graphs of functions $\widehat{\gamma}(y;\xi)$ for various ξ are presented in Fig. 2 (a, b, c, d). In order to be more clear in Fig. 2 c, d, e, f, a part of graph of functions $\widehat{\gamma}(y;\xi)$ and $\widehat{\gamma}_1(y;\xi)$ are hidden in vicinity of a zero. We guaranty that there exists at least one positive eigenfunction to the left of the graph of the function $\widehat{\gamma}(y;\xi)$, in the area T_+. Now we will study the cases when $\gamma<0$ and $\gamma>1$, and $0<\gamma<1$ separately.

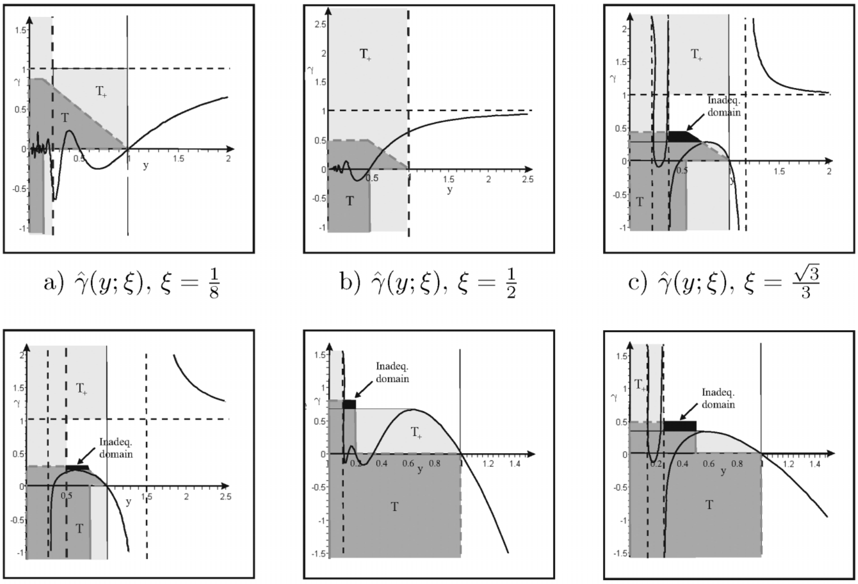
The cases $\gamma<0$ and $\gamma>1$. When parameter of nonlocal boundary condition is $\gamma<0$ or $\gamma>1$, investigated boundary problem has always at least one real positive eigenvalue (see, Fig. 1). If $m\in\mathbb{N}_o$, $n\in\mathbb{N}_e$, there exists constant eigen-



$\xi=\frac{1}{8}$ $\xi=\frac{1}{7}$ $\xi=\frac{1}{2}$

$\xi=\frac{\sqrt{3}}{3}$ $\xi=\frac{3}{4}$ $\xi=1$

Fig. 1. Functions $\gamma(x\pi;\xi)$.



a) $\hat{\gamma}(y;\xi)$, $\xi = \frac{1}{8}$ b) $\hat{\gamma}(y;\xi)$, $\xi = \frac{1}{2}$ c) $\hat{\gamma}(y;\xi)$, $\xi = \frac{\sqrt{3}}{3}$

d) $\hat{\gamma}(y;\xi)$, $\xi = \frac{3}{4}$ e) $\hat{\gamma}_1(y;\xi)$, $\xi = \frac{1}{5}$ f) $\hat{\gamma}_1(y;\xi)$, $\xi = \frac{1}{2}$

Fig. 2. Functions $\widehat{\gamma}(y;\xi)$ and $\widehat{\gamma_1}(y,\xi)$.

value. In these cases, when constant eigenvalues do not exist, the first order poles $p_k = \pi(k-\frac{1}{2})/\xi, k \in \mathbb{N}$ of function $\gamma(x;\xi)$ are important.

Investigating these eigenvalues corresponding positive eigenfunctions, we need to rewiev these cases when constant eigenvalues exist and when do not exist. When parameter $\xi = \frac{m}{n} \in (0,1)$ is such that constant eigenvalues exist and function $\gamma(x;\xi)$ does not have poles, then at least one positive eigenfunction exists in interval $y \in (0;\frac{2}{n})$. In other cases, when function $\gamma(x;\xi)$ has the first order poles $p_k = \pi(k-\frac{1}{2})/\xi, k \in \mathbb{N}$, then at least one positive eigenfunction exists in interval $y \in (0;y_*)$, when $\frac{\pi}{p_1} < 1$ and $y \in (0;1)$, when $\frac{\pi}{p_1} \geqslant 1$. In paper [1] it is said, that when $\gamma < 0$ one positive eigenfunction exists in interval $y \in (0;\xi]$(see, Fig. 2, a, b, c, d, area T). Consequently, when $\gamma < 0$, it is obtained bigger area T_+ of positive eigenfunctions existence. And the area, when $\gamma > 1$, is not investigated by G. Infante in [1].

The case $\mathbf{0} < \gamma < \mathbf{1}$. When $0 < \gamma < 1$, always at least one positive eigenvalue exists (see, Fig. 1). Investigating positive eigenfunctions we will divide into two cases: when $0 < \xi \leqslant \frac{1}{2}$ and $\frac{1}{2} < \xi < 1$. When $0 < \xi \leqslant \frac{1}{2}$, positive eigenfunction exists in interval $y \in (0,1)$ (see, Fig. 2, a) $\xi = \frac{1}{8}$, b) $\xi = \frac{1}{2}$). When $\frac{1}{2} < \xi \leqslant 1$ and $0 < \gamma \leqslant \gamma_*$, as also shown in paper [1], positive eigenfunction exists in interval $y \in (0, 1-\gamma)$. However, when $\gamma_* < \gamma < 1-\xi$, we get inadequacy domain (see, Fig. 2, c) $\xi = \frac{\sqrt{3}}{3}$, d) $\xi = \frac{3}{4}$, black area). G. Infante obtained, that the problem 3–5 has positive eigenfunctions in this area, but the problem 1–2 does not have positive eigenunctions in inadequacy domain. When $\gamma_* < \gamma < 1$, and constant eigenvalues exist then at least one positive eigenfunction exists, when $\gamma \in (0, \frac{2}{n})$. If constant eigenvalues do not exist then at least one positive eigenfunction exists, when $\gamma \in (0, \gamma_*)$, $\gamma_* < \frac{2\xi}{3}$.

In problems 3, 6 and 3, 7 the obtained positive or negative eigenfunctions existence domains coincide with Sturm–Liouville problem 1 with 6 and 7 nonlocal boundary conditions domains. And in problem 3–4 when $\xi_0 < \xi < \xi_1$ ($\xi_0 \approx 0.16\ldots$, $\xi_1 \approx 0.65\ldots$) and $\gamma_* < \gamma < 1 - \xi$, we get inadequacy domain, where positive eigenfunctions do not exist. In this case function $\widehat{\gamma}(y;\xi)$ we denote $\widehat{\gamma}_1(y;\xi)$ (see, Fig. 2, e, f).

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REZIUMĖ

S. Pečiulytė, Štikonas. Apie vieno nelokaliojo kraštinio uždavinio teigiamas tikrines funkcijas

Šiame straipsnyje analizuojamos Šturmo–Liuvilio uždavinio su viena nelokaliąja dvitaške kraštine sąlyga teigiamos tikrinės reikšmės ir tikrinės funkcijos. Gauti rezultatai palyginami su aprašytais G. Infante straipsnyje [1].