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The minimizer for the second order differential problem with one nonlocal condition

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Abstract. In this paper, we investigate the minimizer of the residual for the second order differential problem with one initial and other nonlocal Bitsadze–Samarskii condition. We obtain the representation of the minimizer and present an example.

Keywords: nonlocal conditions, least squares, Moore–Penrose inverse, minimizer.

Introduction

Second order differential problems with nonlocal conditions, various direct representations of the unique solution using Green's functions were obtained by Roman [4]. For example, the unique solution to the problem

$$-u'' = f(x), \qquad x \in [0,1], \tag{1}$$
$$u(0) = g_1, \qquad u'(1) = \gamma u(\xi) + g_2, \tag{2}$$

where $f \in L^2[0,1]$, $g_1, g_2, \gamma \in \mathbb{R}$ and $\xi \in (0,1)$, exists and is given by

$$u = g_1 + \frac{\gamma g_1 + g_2 + \gamma \int_0^1 G(x,y) f(y)\,dy}{1 - \gamma\xi} x + \int_0^1 G(x,y) f(y)\,dy$$

if and only if $\gamma\xi \neq 1$. Here we used the Green's function

$$G(x,y) = \begin{cases} y, & y \leqslant x, \\ x, & y \geqslant x \end{cases} \tag{3}$$

of the problem (1)–(2) with $\gamma = 0$ [4]. If $\gamma\xi = 1$, the problem does not have the unique solution: there may be infinitely many solutions (consistent problem) or no solutions (inconsistent problem). In this paper, we will look for the unique function, which minimizes the residual of the problem (1)–(2) and is "*smallest*" among all minimizers of the residual.

1 Existence of the minimizer

We rewrite the problem (1)–(2) into the equivalent vectorial form

$$\boldsymbol{L}u = \boldsymbol{f}, \tag{4}$$

where $\boldsymbol{f} = (f, g_1, g_2)^\top \in L^2[0,1] \times \mathbb{R}^2$. For the Hilbert space $L^2[0,1] \times \mathbb{R}^2$, we use the inner product $(\boldsymbol{f}, \widetilde{\boldsymbol{f}}) = (f, \widetilde{f})_{L^2[0,1]} + g_1 \cdot \widetilde{g}_1 + g_2 \cdot \widetilde{g}_2$ and the norm

$$\|\boldsymbol{f}\| = \sqrt{\|f\|^2_{L^2[0,1]} + |g_1|^2 + |g_2|^2}.$$

Theorem 1 *The operator $\boldsymbol{L} : H^2[0,1] \to L^2[0,1] \times \mathbb{R}^2$ is continuous and linear with the domain $D(\boldsymbol{L}) = H^2[0,1]$ and the closed range $R(\boldsymbol{L})$.*

Proof. *Domain.* The operator $-u''$ is defined on the whole Sobolev space $H^2[0,1]$. According to the Sobolev embedding theorem [2], every function $u \in H^2[0,1]$ belongs to $C^1[0,1]$. Thus, conditions (2) are defined for every $u \in H^2[0,1]$. It means that the operator $\boldsymbol{L}$ is defined on the whole $H^2[0,1]$.

Linearity. It is obvious that the operator L is linear, since the differential operator (1) and nonlocal conditions (2) are linear with respect to u.

Continuity. From the triangle inequality, we have $\|\boldsymbol{L}u\|^2 = \|u''\|^2_{L_2[0,1]} + |u(0)|^2 + |u'(1) - \gamma u(\xi)|^2 \leqslant C \cdot \|u\|^2_{H^2[0,1]}$, since $\|u\|_{L^2[0,1]} \leqslant \|u\|_{H^2[0,1]}$ and the Sobolev embedding theorem says that for all $u \in H^2[0,1]$ there exist such particular positive constants L_1 and L_2 that

$$|u(0)| \leqslant \max_{x \in [0,1]} |u(x)| \leqslant \|u\|_{C^1[0,1]} \leqslant L_1 \|u\|_{H^2[0,1]},$$
$$|u'(1) - \gamma u(\xi)| \leqslant |u'(1)| + |\gamma| \cdot |u(\xi)| \leqslant (1 + |\gamma|)\|u\|_{C^1[0,1]} \leqslant L_2 \|u\|_{H^2[0,1]}.$$

Precisely, we obtain the estimate $\|\boldsymbol{L}u\| = \left(1 + L_1^2 + L_2^2\right)^{\frac{1}{2}} \|u\|_{H^2[0,1]} \leqslant C\|u\|_{H^2[0,1]}$ for $C = 1 + L_1 + L_2$ and all $u \in H^2[0,1]$, which means that the linear operator $\boldsymbol{L}$ is continuous.

Closeness. Finally, we consider the consistent problem (4). The equation $-u'' = f$ has the general solution

$$u = c_1 + c_2 x + \int_0^1 G(x,y) f(y)\, dy, \qquad c_1, c_2 \in \mathbb{R}.$$

Substituting this general solution to nonlocal conditions, we obtain $c_1 = g_1$ and

$$(1 - \gamma\xi)c_2 = \gamma g_1 + \gamma \int_0^\xi y f(y)\, dy + \gamma\xi \int_\xi^1 f(y)\, dy + g_2.$$

If $\gamma\xi \neq 1$, then we solve c_2 uniquely and obtain the unique solution to the problem (4) with every right hand side $(f, g_1, g_2) \in L^2[0,1] \times \mathbb{R}^2$. It means that the range $R(\boldsymbol{L}) = L^2[0,1] \times \mathbb{R}^2$ is coincident with the whole space and, so, is closed. If $\gamma\xi = 1$, we solve $g_2 = -\gamma g_1 - \gamma \int_0^\xi y f(y)\, dy - \int_\xi^1 f(y)\, dy$ and obtain the representation of the range

$$R(\boldsymbol{L}) = \left\{ (f, g_1, -\gamma g_1 - \gamma \int_0^\xi y f(y)\, dy - \int_\xi^1 f(y)\, dy)^\top \right\} \tag{5}$$

for every $f \in L^2[0,1]$, $g_1 \in \mathbb{R}$. Now we take the sequence $\boldsymbol{f}_n = (f_n, g_{1n}, -\gamma g_{1n} - \gamma \int_0^\xi y f_n(y)\, dy - \int_\xi^1 f_n(y)\, dy)^\top \in R(\boldsymbol{L})$, which converges in the space $L^2[0,1] \times \mathbb{R}^2$, say, to $\boldsymbol{f} = (f, g_1, g_2)^\top \in L^2[0,1] \times \mathbb{R}^2$. Does this limit belong to $R(\boldsymbol{L})$? From the extended form of the limit $\|\boldsymbol{f}_n - \boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^2} \to 0$, if $n \to \infty$, we get $\|f_n - f\|_{L^2[0,1]} \to 0$, $g_{1n} \to g_1$ and $-\gamma g_{1n} - \gamma \int_0^\xi y f_n(y)\,dy - \int_\xi^1 f_n(y)\,dy \to g_2$. Moreover, we obtain

$$
\begin{aligned}
&|\gamma g_1 + \gamma \int_0^\xi y f(y)\,dy + \int_\xi^1 f(y)\,dy - \gamma g_{1n} - \gamma \int_0^\xi y f_n(y)\,dy - \int_\xi^1 f_n(y)\,dy| \\
&\quad \leqslant \gamma |g_{1n} - g_1| + \gamma \int_0^\xi |y(f(y) - f_n(y))| + \int_\xi^1 |f(y) - f_n(y)| dy \\
&\quad \leqslant \gamma |g_{1n} - g_1| + \gamma \|f - f_n\|_{L^2[0,1]} + \|f - f_n\|_{L^2[0,1]} \to 0,
\end{aligned}
$$

if $n \to \infty$. Since the limit is unique, we obtain $g_2 = -\gamma g_1 - \gamma \int_0^\xi y f(y)\,dy - \int_\xi^1 f(y)\,dy$, what means that $(f, g_1, g_2)^\top \in R(\boldsymbol{L})$. □

According to [1], the properties of the operator $\boldsymbol{L}$, collected in Theorem 1, are sufficient to exist functions $u^g \in H^2[0,1]$, those minimize the residual of the problem (4)

$$
\|\boldsymbol{L}u^g - \boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^2} = \min_{u \in H^2[0,1]} \|\boldsymbol{L}u - \boldsymbol{f}\|_{L^2[0,1]\times\mathbb{R}^2}.
$$

Among all those minimizers, there also exists the unique function u^o of the minimum norm, i.e. $\|u^o\|_{H^2[0,1]} < \|u^g\|_{H^2[0,1]}$ for all $u^g \neq u^o$. The minimizer u^o is often called *the minimum norm least squares solution*.

Theorem 2 *The problem* (4) *always has the minimizer* $u^o = \boldsymbol{L}^\dagger \boldsymbol{f}$, *where* $\boldsymbol{L}^\dagger : L^2[0,1] \times \mathbb{R}^2 \to H^2[0,1]$ *is the Moore–Penrose inverse of the operator* $\boldsymbol{L}$.

Proof. It follows from Theorem 1. Precisely, $\boldsymbol{L}$ is the continuous linear operator with $D(\boldsymbol{L}) = H^2[0,1]$ and the closed range. Then according to [1], it has the Moore-Penrose inverse $\boldsymbol{L}^\dagger$, which represents the minimizer by $u^o = \boldsymbol{L}^\dagger \boldsymbol{f}$. □

Since the residual of the consistent problem (1)–(2) is equal to zero, then from the proof of Theorem 1, the general solution to the consistent problem

$$
u^g = g_1 + cx + \int_0^1 G(x,y) f(y)\,dy, \qquad c \in \mathbb{R},
$$

represents all minimizers of the problem with the trivial residual. Minimizing its $H^2[0,1]$ norm or calculating $P_{N(\boldsymbol{L})^\perp} u^g$ [1], we obtain the minimum norm least squares solution

$$
u^o = g_1 + c_o x + \int_0^1 G(x,y) f(y)\,dy, \tag{6}
$$

where $c_o = -\frac{3}{8} g_1 - \frac{3}{4} \int_0^1 \big(x \int_0^1 G(x,y) f(y)\,dy + \int_x^1 f(y)\,dy\big) dx$.

How to represent the minimizer to the inconsistent problem, which has no solutions? First, we need to discuss about the discretized problem and its minimizer.

2 The discrete minimizer

Now we introduce the mesh $\overline{\omega}^h := \{x_i = ih,\ nh = 1,\ i = 0, 1, \dots, n\}$ for $n \in \mathbb{N}$ and suppose ξ is coincident with the mesh point, i.e. $\xi = sh$ for $s \in \{1, 2, \dots, n-2\}$. Then we discretize the problem (1)–(2) as follows

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i, \quad i = \overline{1, n-1}, \tag{7}$$

$$u_0 = g_1, \qquad \frac{u_n - u_{n-1}}{h} = \gamma u_s + g_2, \tag{8}$$

which can be rewritten in the equivalent matrix form $\mathbf{Au} = \mathbf{b}$ for the matrix $\mathbf{A} \in \mathbb{R}^{(n+1)\times(n+1)}$ and the right hand side $\mathbf{b} = (f_1, f_2, \dots, f_{n-1}, g_1, g_2)^\top \in \mathbb{R}^{(n+1)\times 1}$, where we can take $f_i = f(x_i)$ if $f \in C[0,1]$. For every $\mathbf{u},\ \mathbf{b} \in \mathbb{R}^{(n+1)\times 1}$ introducing two discrete norms

$$\|\mathbf{u}\|_{H^2(\overline{\omega}^h)} = \left(\sum_{i=0}^{n} u_i^2 h + \sum_{i=1}^{n}\left(\frac{u_i - u_{i-1}}{h}\right)^2 h + \sum_{i=1}^{n-1}\left(\frac{u_{i+1} - u_i + u_{i-1}}{h^2}\right)^2 h\right)^{1/2},$$

$$\|\mathbf{b}\|_{L^2(\overline{\omega}^h)\times\mathbb{R}^2} = \left(\sum_{i=0}^{n-2} b_i^2 h + |b_{n-1}|^2 + |b_{n-2}|^2\right)^{1/2}$$

we consider the operator $\mathbf{A} : H^2(\overline{\omega}^h) \to L^2(\overline{\omega}^h) \times \mathbb{R}^2$. According to [1], there exists its Moore-Penrose inverse $\mathbf{A}^\dagger : L^2(\overline{\omega}^h) \times \mathbb{R}^2 \to H^2(\overline{\omega}^h)$, which represents the minimizer $\mathbf{u}^o = \mathbf{A}^\dagger \mathbf{b} \in \mathbb{R}^{(n+1)\times 1}$ of the discretized problem (7)–(8).

3 Minimizer of the inconsistent problem

Theorem 3 *The minimizer u^o of the differential problem* (1)–(2) *is equal to the minimizer of the consistent problem*

$$\boldsymbol{L}u = \boldsymbol{f} - \frac{(\boldsymbol{v}, \boldsymbol{f})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}, \tag{9}$$

where

$$\boldsymbol{v}(x) = \left(\begin{cases}\gamma x, & x \leqslant \xi,\\ 1, & x \geqslant \xi,\end{cases} \gamma, 1\right)^\top \in R(\boldsymbol{L})^\perp. \tag{10}$$

Remark 1. If the problem (4) is consistent, i.e. $\boldsymbol{f} \in R(\boldsymbol{L})$, then $(\boldsymbol{v}, \boldsymbol{f}) = 0$ and we obtain the same consistent problem (4).

Proof. Let us consider the problem (1)–(2) with $\gamma\xi = 1$. The minimizer u^o to the problem $\boldsymbol{L}u = \boldsymbol{f}$ is the minimizer to the consistent problem $\boldsymbol{L}u = \mathbf{P}_{R(\boldsymbol{L})}\boldsymbol{f}$ [1]. Thus, u^o is the exact solution to this consistent problem. Similarly, the minimizer $\mathbf{u}^o$ of the discretized problem $\mathbf{Au} = \mathbf{b}$ is the minimizer to the consistent problem $\mathbf{Au} = \mathbf{P}_{R(\mathbf{A})}\mathbf{b}$. We obtain the general solution to the discrete consistent problem

$$u_i^{h,g} = g_1 + cx_i + \sum_{j=0}^{n-2} G_{ij} b_j h, \qquad c \in \mathbb{R}, \tag{11}$$

where

$$G_{ij}=\begin{cases}x_{j+1}, & j\leqslant i,\\ x_i, & j\geqslant i\end{cases}$$

is the discrete Green's function to the problem (7)–(8) with $\gamma=0$ [4]. For simplicity, we suppose $f\in C[0,1]$, take $f_i=f(x_i)$ and $\mathbf{f}=(f(x_0),f(x_1),\dots,f(x_{n-2}))$. Substituting (11) into (8), we get the representation $g_2=-\gamma g_1-\gamma\sum_{i=1}^{s}x_if_ih-\sum_{i=s+1}^{n-1}f_ih$, which can be rewritten as follows $g_2=-\gamma g_1-\gamma\int_0^\xi xf(x)\,dx-\int_\xi^1 f(x)\,dx+O(h)$. So, now the range of the discrete problem is given by

$$R(\mathbf{A})=\left\{\left(\mathbf{f},g_1,-\gamma g_1-\gamma\sum_{i=1}^{s}x_if(x_i)h-\sum_{i=s+1}^{n-1}f(x_i)h\right)^\top\right\},$$

which, as we observe, represents the discretization of (5). Precisely, if $\mathbf{b}\in R(\mathbf{A})$, then $\mathbf{b}=\mathbf{f}^h+\mathbf{O}$, where $\mathbf{f}^h=(\mathbf{f},g_1,-\gamma g_1-\gamma\int_0^\xi xf(x)\,dx-\int_\xi^1 f(x)\,dx)^\top$ is the discretization of $\boldsymbol{f}$ and $O_i=O(h)$ for $i=\overline{0,n}$. In general for $\mathbf{b}\notin R(\mathbf{A})$, we get $\widetilde{\mathbf{b}}=\mathbf{P}_{R(\mathbf{A})}\mathbf{b}=\mathbf{P}_{N(\mathbf{A}^*)^\perp}\mathbf{b}=\mathbf{b}-\mathbf{P}_{N(\mathbf{A}^*)}\mathbf{b}$. According to [3], $\dim N(\mathbf{A}^*)=1$ ant the nullspace of the adjoint matrix $N(\mathbf{A}^*)$ is composed of the one vector

$$\mathbf{v}=\left(\begin{cases}\gamma x_{i+1}, & i+1\leqslant s,\\ 1, & i+1\geqslant s,\end{cases}\ \gamma,1\right)^\top\in\mathbb{R}^{(n+1)\times 1}\quad (i=\overline{0,n-2}),$$

which converges to the function (10). The limit function $\boldsymbol{v}\in R(\boldsymbol{L})^\perp$, because taking $\boldsymbol{f}\in R(\boldsymbol{L})$, we get $(\boldsymbol{v},\boldsymbol{f})=0$. Then $\mathbf{P}_{R(\mathbf{A})}\mathbf{b}=\mathbf{b}-\frac{\mathbf{v}}{\|\mathbf{v}\|^2}(\mathbf{v},\mathbf{b})=\widetilde{\mathbf{f}}^h+\mathbf{O}$. We can directly verify that $\widetilde{\boldsymbol{f}}=\boldsymbol{f}-\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|^2}(\boldsymbol{v},\boldsymbol{f})\in R(\boldsymbol{L})$ for every $f\in L^2[0,1]$, since it is of the corresponding form (5). It means that the problem $\boldsymbol{L}u=\widetilde{\boldsymbol{f}}$ is consistent. This problem has the minimizer

$$\widetilde{u}^o=\boldsymbol{L}^\dagger\widetilde{\boldsymbol{f}}=\boldsymbol{L}^\dagger\left(\boldsymbol{f}-\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|^2}(\boldsymbol{v},\boldsymbol{f})\right)=\boldsymbol{L}^\dagger\boldsymbol{f}-\frac{(\boldsymbol{v},\boldsymbol{f})}{\|\boldsymbol{v}\|^2}\boldsymbol{L}^\dagger\boldsymbol{v}=\boldsymbol{L}^\dagger\boldsymbol{f}=u^o,$$

which is equal to the minimizer u^o of the problem $\boldsymbol{L}u=\boldsymbol{f}$, since $\boldsymbol{v}\in R(\boldsymbol{L})^\perp=N(\boldsymbol{L}^*)=N(\boldsymbol{L}^\dagger)$ [1]. □

Since the problem (9) is consistent, its minimizer is of the corresponding form (6). Then expanding the right hand side of (9) and substituting

$$g_1-\gamma\frac{(\boldsymbol{v},\boldsymbol{f})}{\|\boldsymbol{v}\|^2}\quad\text{and}\quad f(x)-\frac{(\boldsymbol{v},\boldsymbol{f})}{\|\boldsymbol{v}\|^2}\begin{cases}\gamma x, & x\leqslant\xi,\\ 1, & x\geqslant\xi\end{cases}$$

instead of g_1 and $f(x)$, respectively, in (6), we obtain the representation of the minimizer u^o for the (in)consistent problem (1)–(2). The obtained representation of u^o simplifies to (6) if the problem (4) is consistent, because $(\boldsymbol{v},\boldsymbol{f})=0$. The residual of the problem (1)–(2) is equal to $\boldsymbol{r}(u):=\boldsymbol{L}u-\boldsymbol{f}$. So,

$$\boldsymbol{r}(u^o)=\boldsymbol{L}u^o-\boldsymbol{f}=\boldsymbol{f}-\frac{(\boldsymbol{v},\boldsymbol{f})}{\|\boldsymbol{v}\|^2}\boldsymbol{v}-\boldsymbol{f}=-\frac{(\boldsymbol{v},\boldsymbol{f})}{\|\boldsymbol{v}\|^2}\boldsymbol{v},\quad \|\boldsymbol{r}(u^o)\|_{L^2[0,1]\times\mathbb{R}^2}=\frac{|(\boldsymbol{v},\boldsymbol{f})|}{\|\boldsymbol{v}\|}.$$

Example. Let us now consider the particular problem (1)–(2) with $f(x) = 0$, $x \in [0, 1]$, $g_1 = 0$, $g_2 = 1$, $\gamma = 2$ and $\xi = 1/2$. We can easily verify that this problem has no solutions and find the least minimizer u^o of the residual. Let us note that the residual $\|\boldsymbol{r}(u^o)\| \leqslant \|\boldsymbol{r}(x)\| = 1$. First, we calculate the right hand side of the consistent problem (9)

$$\widetilde{\boldsymbol{f}} = \boldsymbol{f} - \frac{(\boldsymbol{v}, \boldsymbol{f})}{\|\boldsymbol{v}\|^2}\boldsymbol{v} = -\frac{1}{17}\left(\begin{cases} 6x, & x \leqslant 1/2, \\ 3, & x \geqslant 1/2 \end{cases}, 6, -14\right)^\top.$$

Substituting the first two components instead of $f(x)$ and g_1 into (6), we get the minimizer

$$u^o = -\frac{6}{17} + \frac{3 \cdot 1959}{8 \cdot 17 \cdot 640}x + \frac{1}{8 \cdot 17}\begin{cases} 8x^3 - 18x, & x \leqslant 1/2, \\ 12x^2 - 24x + 1, & x \geqslant 1/2. \end{cases}$$

As we predicted the norm of the minimal residual $\|\boldsymbol{r}(u^o)\| = \sqrt{3/17}$ is less than the residual with the function x, i.e. $\|\boldsymbol{r}(x)\| = 1$. We can easily verify that the obtained $u^o \in H^2[0, 1]$.

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REZIUMĖ

Antrosios eilės diferencialinio uždavinio su viena nelokaliąja sąlyga minimalusis sprendinys

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Šiame darbe nagrinėsime antrosios eilės diferencialinio uždavinio su viena pradine ir kita nelokaliąja sąlygomis netiktį minimizuojančią funkciją. Gausime šios minimizuojančios funkcijos išraišką ir pateiksime pavyzdį.

Raktiniai žodžiai: nelokalios sąlygos, mažiausi kvadratai, Moore–Penrose atvirkštinis atvaizdis, minimizuojanti funkcija