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Nullspace of the m-th order discrete problem with nonlocal conditions*

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Abstract. In this paper, we obtain the representation of the nullspace for m-th order discrete linear problems with nonlocal conditions. An example is given.

Keywords: discrete problem, nonlocal conditions, nullspace.

Introduction

In paper [1], we considered a method how to calculate directly generalized discrete Green's function for the second order discrete problem with two nonlocal conditions. The basic part of the investigation was played by the solution to the nullspace of the discrete problem.

Let us introduce the set $X_n := \{0, 1, 2, \dots, n\}$ and the space of complex functions $F(X_n) := \{u | u : X_n \to \mathbb{C}\}$. In this paper, we are going to represent the nullspace for the m-th order discrete linear problem with m nonlocal conditions

$$(\mathcal{L}u)_i := a_i^m u_{i+m} + \dots + a_i^1 u_{i+1} + a_i^0 u_i = f_i, \quad a_i^0, a_i^m \neq 0, \quad i \in X_{n-m}, \tag{1}$$

$$\langle L_k, u \rangle := \sum_{j=0}^{n} L_k^j u_j = g_k, \quad k = \overline{1, m}, \tag{2}$$

where $a^j \in F(X_{n-m})$, $j = \overline{0, m}$, $f \in F(X_{n-m})$, $L_k \in F^*(X_n)$, $g_k \in \mathbb{C}$ for $k = \overline{1, m}$ and $n \geqslant m$.

1 Equivalent matrix problem

The solution $u \in F(X_n)$ is uniquely represented by the column matrix $\mathbf{u} = (u_0, u_1, \dots, u_n)^T \in \mathbb{C}^{(n+1)\times 1}$ but every discrete linear functional $L_k \in F^*(X_n)$ is described by a complex row matrix $\mathbf{L}_k = (L_k^0, L_k^1, \dots, L_k^n) \in \mathbb{C}^{1\times(n+1)}$, $k = \overline{1, m}$. Similarly, the operator $\mathcal{L} : F(X_n) \to F(X_{n-m})$ is uniquely represented by the matrix $\mathbf{L} = (\mathcal{L}_{ij}) \in \mathbb{C}^{(n-m+1)\times(n+1)}$, which has rows $\mathbf{L}_{i\cdot} = (0, \dots, 0, a_i^0, a_i^1, \dots, a_i^m,$

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$0,\dots,0) \in \mathbb{C}^{1\times(n+1)}$, where the element a_i^0 follows after i zeroes for each $i \in X_{n-m}$. Thus, the problem (1)–(2) can be rewritten in the equivalent matrix form $\mathbf{A}\mathbf{u} = \mathbf{b}$ with $\mathbf{b} = (f_0, f_1, \dots, f_{n-m}, g_1, \dots, g_m)^T \in \mathbb{C}^{(n+1)\times 1}$ and $\mathbf{A} = (\mathbf{L}, \mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_m)^T \in \mathbb{C}^{(n+1)\times(n+1)}$.

2 $N(\mathbf{A})$ basis

Let us now focus our investigation on the homogeneous problem

$$\mathbf{A}\mathbf{u} = \mathbf{0}. \tag{3}$$

First, we note that $\mathbf{L}$ is the trapezoid matrix with nonzero diagonal elements a_0^i, $i \in X_{n-m}$. So, it has the full row rank, precisely, $\operatorname{rank}\mathbf{L} = n-m+1$. Then the nullity $d := \dim N(\mathbf{A}) \in \{0,1,2,\dots,m\}$. The nullspace $N(\mathbf{A})$ of the problem (3) is nonzero if the matrix $\mathbf{A}$ is singular, or equivalently [2]

$$D(\boldsymbol{L})[\boldsymbol{u}] := \begin{vmatrix} \langle L_1, u^1\rangle & \langle L_2, u^1\rangle & \dots & \langle L_m, u^1\rangle \\ \dots & \dots & \dots & \dots \\ \langle L_1, u^m\rangle & \langle L_2, u^m\rangle & \dots & \langle L_m, u^m\rangle \end{vmatrix} = 0 \tag{4}$$

for every fundamental system $\boldsymbol{u} = (u^1, u^2, \dots, u^m)$ of the equation (1). Here we denoted the collection of functionals $\boldsymbol{L} = (L_1, L_2, \dots, L_m)$. If $\dim N(\mathbf{A}) = m$, then the nullspace of problem the (3) is coincident with u^k, $k = \overline{1,m}$, – the fundamental system of (1). Let us find the representation of the nullspace if $0 < d < m$.

As in paper [1], the rank of matrix $\mathbf{A}$ is $\operatorname{rank}\mathbf{A} = n+1-d$. Thus, there are d functionals $\mathbf{L}_{k_j}$, $j = \overline{1,d}$, those are linear combinations of other (linearly independent) rows of $\mathbf{A}$, representing the operator $\mathcal{L}$ and rest functionals $\mathbf{L}_{k_j}$, $j = \overline{d+1,m}$. Moreover, the matrix $\mathbf{A}$ has linearly independent columns with indexes $s_0 = 0$, $s_1 = 1, \dots, s_{n-m} = n-m, \dots, s_{n-d}$. Other columns with indexes s_{n-d+j}, $j = \overline{1,d}$, are linear combinations of them. We note that $s_i = i$, $i \in X_{n-m}$, and $s_{n-m+j} \in \{n-m+1, \dots, n\}$, $j = \overline{1,m}$.

As in [1], we introduce the corresponding auxiliary nonsingular problem

$$\begin{aligned} (\mathcal{L}\omega)_i &= g_i^1, \quad i \in X_{n-m}, \\ \langle l_{k_j}, \omega\rangle &:= \omega_{s_{n-d+j}} = 0, \quad j = \overline{1,d}, \\ \langle l_{k_j}, \omega\rangle &:= \langle L_{k_j}, \omega\rangle = g^1_{n-m-d+j}, \quad j = \overline{d+1,m}, \end{aligned} \tag{5}$$

where $\omega \in F(X_n)$ and $\mathbf{g}^1 = \mathbf{g}^1(u_{s_{n-d+1}}, \dots, u_{s_n}) \in \mathbb{C}^{(n-d+1)\times 1}$ is of the form

$$g_i^1 = -\sum_{l=1}^{d} u_{s_{n-d+l}} \begin{cases} \mathcal{L}_{i,s_{n-d+l}}, & i \in X_{n-m}, \\ L^{s_{n-d+l}}_{k_{m-n+d+i}}, & i = \overline{n-m+1, n-d}. \end{cases} \tag{6}$$

Let us denote the matrix of this auxiliary problem (5) by $\widetilde{\mathbf{A}}$. Similarly as in [1], we derive the solution of (3), which is given below

$$u_{s_i} = \omega_{s_i} = \sum_{j=0}^{n-m} G_{s_i,j} g_j^1 + \sum_{j=d+1}^{m} g^1_{n-m-d+j} v_{s_i}^{k_j}, \quad i \in X_{n-d}, \tag{7}$$

and u_{s_i}, $i=\overline{n-d+1,n}$, are arbitrary complex numbers. Here $G \in F(X_n \times X_{n-m})$ is ordinary discrete Green's function of the problem (6), since we constructed the problem (6) with the nonsingular matrix [1]. Moreover, v^k, $k=\overline{1,m}$, form the biorthogonal fundamental system of the problem (6) [2]. For example, taking $u_{s_{n-d+l}}$, $l=\overline{1,d}$, such as $u_{s_{n-d+l}} \cdot u_{s_{n-d+j}} = \delta_{lj}$, $l,j=\overline{1,d}$, we get $g_i^1 = \mathcal{L}_{i,s_{n-d+l}}$, $i \in X_{n-m}$, and $g_i^1 = L_{k_{m-n+d+i}}^{s_{n-d+l}}$, $i=\overline{n-m+1,n-d}$, for every fixed $l=\overline{1,d}$. Here δ_{ij} is the Kronecker delta.

Thus, we obtain the particular basis of the nullspace w^l, $l=\overline{1,d}$, for the problem (1)–(2), where (7) simplifies to

$$w_{s_i}^l = -\sum_{j=0}^{n-m} G_{s_i,j}\mathcal{L}_{j,s_{n-d+l}} - \sum_{j=n-m+1}^{n-d} v_{s_i}^{k_{m-n+d+j}} L_{k_{m-n+d+j}}^{s_{n-d+l}}, \quad i \in X_{n-d}. \tag{8}$$

The equality $\mathbf{I} = \widetilde{\mathbf{A}}^{-1}\widetilde{\mathbf{A}}$, where $\mathbf{I} = (\delta_{ij}) \in \mathbb{C}^{(n+1)\times(n+1)}$ is the identity matrix, is given by $\mathbf{I} = (\mathbf{G}, \mathbf{v}^1, \dots, \mathbf{v}^m)(\mathbf{L}, \mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_m)^T$. Rewriting it in the discrete form, we have $\delta_{ij} = \sum_{p=0}^{n-m} G_{ip}\mathcal{L}_{pj} + \sum_{k=1}^{m} v_i^k L_k^j$, $i,j \in X_n$. We apply this equality to (8), use $\delta_{s_i} \cdot \delta_{s_{n-d+l}} = 0$ for $i \in X_{n-d}$ and $l=\overline{1,d}$, and obtain

$$w_{s_i}^l = \sum_{j=1}^{d} v_{s_i}^{k_j} L_{k_j}^{s_{n-d+l}}, \quad i \in X_{n-d},\ l=\overline{1,d}. \tag{9}$$

We remember that $w_{s_i}^l \cdot w_{s_j}^l = \delta_{ij}$, $i,j=\overline{n-d+1,n}$, and get the nullspace of the problem (1)–(2), spanned by d linearly independent vectors

$$\mathbf{w}^l = \sum_{i=0}^{n-d}\sum_{j=1}^{d} v_{s_i}^{k_j} L_{k_j}^{s_{n-d+l}} \mathbf{e}^{s_i} + \mathbf{e}^{s_{n-d+l}}, \quad l=\overline{1,d}. \tag{10}$$

3 $N(\mathbf{A}^*)$ basis

Applying literally the proof from [1], we find the nullspace $\widetilde{\mathbf{w}}^l$ of the adjoint matrix $\mathbf{A}^*$, where

$$\widetilde{\mathbf{w}}^l = -\sum_{i=0}^{n-m} \overline{\langle L_{k_l}, G_{\cdot i}\rangle}\mathbf{e}^i - \sum_{i=d+1}^{m} \overline{\langle L_{k_l}, v^{k_i}\rangle}\mathbf{e}^{n-m+k_i} + \mathbf{e}^{n-m+k_l}, \quad l=\overline{1,d}. \tag{11}$$

Theorem 1 *The problem* (1)–(2) *with a singular matrix is solvable if and only if the following orthogonality conditions are valid*:

$$\sum_{i=0}^{n-m} \overline{\langle L_{k_l}, G_{\cdot i}\rangle} f_i + \sum_{i=d+1}^{m} \overline{\langle L_{k_l}, v^{k_i}\rangle} g_{k_i} = g_{k_l}, \quad l=\overline{1,d}.$$

4 Applications to a particular problem

Let us now consider the m-th order differential problem with one Bicadze–Samraskii condition

$$u^{(m)} = f(x), \quad x \in [0,1], \tag{12}$$

$$u(0) = \widetilde{g}_1, \quad u'(0) = \widetilde{g}_2, \quad \dots, \quad u^{(m-2)}(0) = \widetilde{g}_{m-1}, \tag{13}$$

$$u(1) - \gamma u(\xi) = \widetilde{g}_m, \tag{14}$$

for $\xi \in (0,1)$, all γ, $\widetilde{g}_j \in \mathbb{R}$, $j = \overline{1,m}$, and real $f \in C[0,1]$. Let us introduce the mesh $\overline{\omega}^h := \{x_i : x_i = ih,\ nh = 1\}$ and suppose ξ is coincident with a mesh point. i.e. $\xi = sh$ for $s = \overline{1, n-1}$. We also denote $f_i = f(x_{i+1})h^m$ and $g_k = h^{k-1}\widetilde{g}_k$ for $k = \overline{1, m-1}$, and $g_k = \widetilde{g}_k$. Let us introduce finite differences $\nabla^0 u_i = u_i$, $\nabla u_i = \nabla^1 u_i = u_{i+1} - u_i$ and $\nabla^{k+1} u_i = \nabla(\nabla^k u)_i$ for $k \geqslant 0$. Then we apply the finite difference method on the uniform grid $\overline{\omega}^h$ and obtain the m-th order discrete problem

$$(\mathcal{L}u)_i := \nabla^m u_i = f_i, \quad i \in X_{n-m}, \tag{15}$$

$$\langle L_k, u\rangle := \nabla^{k-1} u_0 = g_k, \quad k = \overline{1, m-1}, \tag{16}$$

$$\langle L_m, u\rangle := u_n - \gamma u_s = g_m, \tag{17}$$

that can be represented by a linear system $\mathbf{A}\mathbf{u} = \mathbf{b}$, introduced in Section 1, with the matrix $\mathbf{A} = \mathbf{A}(\gamma)$, $\gamma \in \mathbb{R}$.

First, we consider the classical problem (15)–(17) with $\gamma = 0$ and the matrix $\widetilde{\mathbf{A}} = \mathbf{A}(0)$. Then applying the additivity property of a column of the determinant and remembering that the determinant with two equal columns is equal to zero, we rewrite the condition (4) as follows

$$\begin{aligned} D^{cl} :&= D(L_1, L_2, \dots, L_{m-2}, \delta_n)[\boldsymbol{u}] = D(\delta_0, \delta_1 - \delta_0, L_3, \dots, L_{m-2}, \delta_n)[\boldsymbol{u}] \\ &= D(\delta_0, \delta_1, L_3, \dots, L_{m-2}, \delta_n)[\boldsymbol{u}] - D(\delta_0, \delta_0, L_3, \dots, L_{m-2}, \delta_n)[\boldsymbol{u}] \\ &= D(\delta_0, \delta_1, L_3, \dots, L_{m-2}, \delta_n)[\boldsymbol{u}] = \dots = D(\delta_0, \delta_1, \delta_2, \dots, \delta_{m-2}, \delta_n)[\boldsymbol{u}], \end{aligned}$$

were $\langle \delta_i, u\rangle := u_i,\ i \in X_n$. Furthermore,

$$D^{cl} = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & h & 2h & \dots & (m-2)h & 1 \\ 0 & h^2 & (2h)^2 & \dots & \bigl((m-2)h\bigr)^2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & h^{m-1} & (2h)^{m-1} & \dots & \bigl((m-2)h\bigr)^{m-1} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} h & 2h & \dots & (m-2)h & 1 \\ h^2 & (2h)^2 & \dots & \bigl((m-2)h\bigr)^2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ h^{m-1} & (2h)^{m-1} & \dots & \bigl((m-2)h\bigr)^{m-1} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} h & 2h & \dots & (m-2)h & 1 \\ h^2-h & (2h)^2-2h & \dots & \bigl((m-2)h\bigr)^2-(m-2)h & 0 \\ \dots & \dots & \dots & \dots & \dots \\ h^{m-1}-h^{m-2} & (2h)^{m-1}-(2h)^{m-1} & \dots & \bigl((m-2)h\bigr)^{m-1}-\bigl((m-2)h\bigr)^{m-2} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} h & 2h & \dots & (m-2)h & 1 \\ h(h-1) & 2h(2h-1) & \dots & (m-2)h\bigl((m-2)h-1\bigr) & 0 \\ \dots & \dots & \dots & \dots & \dots \\ h^{m-2}(h-1) & (2h)^{m-2}(2h-1) & \dots & \bigl((m-2)h\bigr)^{m-2}\bigl((m-2)h-1\bigr) & 0 \end{vmatrix}$$

$$= (-1)^m \begin{vmatrix} h(h-1) & 2h(2h-1) & \dots & (m-2)h\bigl((m-2)h-1\bigr) \\ h^2(h-1) & (2h)^2(2h-1) & \dots & ((m-2)h)^2\bigl((m-2)h-1\bigr) \\ \dots & \dots & \dots & \dots \\ h^{m-2}(h-1) & (2h)^{m-2}(2h-1) & \dots & \bigl((m-2)h\bigr)^{m-2}\bigl((m-2)h-1\bigr) \end{vmatrix}$$

$$= (-1)^{m-1}\prod_{k=0}^{m-2}(kh-1) \begin{vmatrix} h & 2h & \dots & (m-2)h \\ h^2 & (2h)^2 & \dots & ((m-2)h)^2 \\ \dots & \dots & \dots & \dots \\ h^{m-2} & (2h)^{m-2} & \dots & \bigl((m-2)h\bigr)^{m-2} \end{vmatrix}$$

$$= (-1)^{m-1}\prod_{k=0}^{m-2}(kh-1) \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & h & 2h & \dots & (m-2)h \\ 0 & h^2 & (2h)^2 & \dots & ((m-2)h)^2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & h^{m-2} & (2h)^{m-2} & \dots & \bigl((m-2)h\bigr)^{m-2} \end{vmatrix},$$

where all $kh-1 \neq 0$, because $kh \leqslant (m-2)h \leqslant (n-2)h < nh = 1$ for every $k = \overline{0, m-2}$. From here follows that the classical problem (15)–(17) with $\gamma = 0$ always has the unique exact solution, since the existence condition (4) of the unique exact solution is always fulfilled:

$$D^{cl} = (-1)^{m-1}\prod_{k=0}^{m-2}(kh-1) D(\delta_0, \delta_1, \dots, \delta_{m-2})\bigl[1, x, \dots, x^{m-2}\bigr] \neq 0.$$

Here $D(\delta_0, \delta_1, \dots, \delta_{m-2})[1, x, \dots, x^{m-2}] \neq 0$ is the condition (4) for the corresponding $(m-1)$-th order problem (15) with initial conditions, which always has the unique solution [2]. Thus, $\det \widetilde{\mathbf{A}} \neq 0$.

Now we note that the condition (4) for the problem (15)–(17) with any real γ is of the form

$$D(\boldsymbol{L})[\boldsymbol{u}] := D^{cl} - \gamma D(\delta_0, \delta_1, \dots, \delta_{m-2}, \delta_s)[\boldsymbol{u}] = D^{cl}\bigl(1 - \gamma v_s^m\bigr), \tag{18}$$

where $v_i^m := D(\delta_0, \delta_1, \dots, \delta_{m-2}, \delta_i)[\boldsymbol{u}]/D^{cl}$, $i \in X_n$, is the unique exact solution to the classical problem $\mathcal{L}u = 0$, $\langle L_j, u\rangle = 0$, $j = \overline{1, m-1}$, $u_n = 1$ [2]. So, (18) equals to zero if and only if $\gamma v_s^m = 1$. This condition is equivalent to $\det \mathbf{A} = 0$.

Let us find the nullspaces to the problem (15)–(17) with the singular matrix, i.e. $\det \mathbf{A} = 0$ or, equivalently, $\gamma v_s^m = 1$.

First, we have $d = \dim N(\mathbf{A}) = 1$, because all rows of the nonsingular matrix $\widetilde{\mathbf{A}}$ are linearly independent, and the singular $\mathbf{A}$ differs from $\widetilde{\mathbf{A}}$ with the last row only.

From here also follows that the functional $\mathbf{L}_m$ is a linear combination of other rows of $\mathbf{A}$, representing the operator $\mathcal{L}$ and functionals $\mathbf{L}_k$, $k = \overline{1, m-1}$. Thus, $k_1 = m$ and $k_j = j - 1$, $j = \overline{2, m}$.

Calculating the determinant with respect to the last row, i.e. $\det \widetilde{\mathbf{A}} = M_n \neq 0$, we obtain that the first n-th order minor M_n of the $(n+1)$-th order matrix $\widetilde{\mathbf{A}}$ is nonzero. It follows that the matrix $\mathbf{A}$ also has the same first n-th order nonzero minor M_n, since matrices $\mathbf{A}$ and $\widetilde{\mathbf{A}}$ differs only with the last row. Then first n columns of $\mathbf{A}$ are linearly independent. They have indices $s_i = i$, $i \in X_{n-1}$. The last column, which has the index $s_n = n$, is a linear combination of them because $\dim N(\mathbf{A}) = 1$.

Now we note that the auxiliary problem (5) for problem (15)–(17) is coincident with the classical problem (15)–(17) ($\gamma = 0$). Since $D^{cl} \neq 0$, this classical problem has ordinary discrete Green's function G^{cl}_{ij}, $i \in X_n$, $j \in X_{n-m}$, and the ordinary biorthogonal fundamental system v^k, $k = \overline{1, m}$. Thus, $G = G^{cl}$. Simplifying (10), we get $\mathbf{w} = \mathbf{v}^m \in N(\mathbf{A})$, which spans the nullspace of $\mathbf{A}$. Moreover, the nullspace $N(\mathbf{A}^*)$ is spanned by the nonzero vector (11), given below

$$\widetilde{\mathbf{w}} = \gamma \sum_{j=0}^{n-m} G^{cl}_{sj}\mathbf{e}^j + \gamma \sum_{j=n-m+1}^{n-1} v_s^{j-n+m}\mathbf{e}^j + \mathbf{e}^n \in N\big(\mathbf{A}^*\big).$$

According to [2], we can always find explicit representations of functions G^{cl} and v^k, $k = \overline{1, m}$.

Corollary 2 *The problem* (15)–(17) *has a solution if and only if*

$$\gamma \sum_{i=0}^{n-m} G^{cl}_{si} f_i + \gamma \sum_{k=1}^{m-1} g_k v_s^k + g_m = 0.$$

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REZIUMĖ

m-osios eilės diskrečiojo uždavinio su nelokaliosiomis sąlygomis nulių aibė

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Šiame darbe yra gautas m-osios eilės diskečiojo tiesinio uždavinio su nelokaliosiomis sąlygomis nulių aibės pavidalas. Pateiktas pavyzdys.

Raktiniai žodžiai: diskretusis uždavinys, nelokaliosios sąlygos, nulių aibė.