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Discounted payments theorems for large deviations

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Abstract. Let $Z(t)=\sum_{j=1}^{N(t)} X_j$, $t\geqslant 0$, be a stochastic process, where X_j are independent identically distributed random variables, and $N(t)$ is non-negative integer-valued process with independent increments. Throughout, we assume that $N(t)$ and X_j are independent. The paper considers normal approximation to the distribution of properly centered and normed random variable $Z_\delta=\int_0^\infty e^{-\delta t}\,dZ(t)$, $\delta>0$, taking into consideration large deviations both in the Cramér zone and the power Linnik zones. Also, we obtain a nonuniform estimate in the Berry–Essen inequality.

Keywords: cumulant method, large deviations, nonuniform estimate, the sum of discounted payments.

Introduction

Let $\{X, X_j,\ j=1,2,\dots\}$ be a family of independent identically distributed (i.i.d.) random variables (r.v.s.) with the mean, variance and distribution function

$$\mu=\mathbf{E}X,\qquad \sigma^2=\mathbf{D}X<\infty,\qquad F_X(x)=\mathbf{P}(X<x),\quad x\in\mathbb{R},$$

where $\mathbb{R}$ is the set of real numbers. In addition, assume that $N(t)$ is non-negative, integer-valued process, with independent increments: for any t_j, $j=0,1,\dots,n$, $n\geqslant 1$, such that $0=t_0<t_1<\dots<t_n<\infty$, the increments $N(t_1)-N(t_0),\dots,N(t_n)-N(t_{n-1})$, are independent. The mean, variance and distribution of $N(t)$ are denoted by

$$\alpha(t)=\mathbf{E}N(t),\quad \beta^2(t)=\mathbf{D}N(t),\quad \mathbf{P}(N(t)=m)=q_m,\quad 0<q_m<1,\ m\in\mathbb{N}_0,\quad (1)$$

where $\mathbb{N}_0=\{0,1,2,\dots\}$. By

$$Z(t)=\sum_{j=1}^{N(t)} X_j,\quad t\geqslant 0,\qquad (2)$$

we denote a stochastic process, where X_j is independent of $N(t)$. For instance, in the continuous dynamic models of an insurance stock [6, p. 152], $R(t)=R(0)+P(t)-Z(t)$, $t\geqslant 0$, can express the surplus $R(t)$ at time t. Here $R(0)$ is the initial reserve and $P(t)$ is the total premium received up to time t. That is, the company sells insurance policies and receives a premium according to $P(t)$. The sum (2) is the total claim amount process in the time interval $[0,t]$. In this example, X_j, $j=1,2,\dots$, denotes the jth claim, and $N(t)$ is the number of claims by time t.

In this paper, we consider random variable (r.v.)

$$Z_\delta = \int_0^\infty e^{-\delta t}\, dZ(t), \tag{3}$$

where $\delta > 0$. For instance, (3) represents the sum of the discounted payments, where is a rate of interest (see, e.g. [2, 8]).

Since $N(t)$ is independent of X_j, $j = 1, 2, \ldots$, the mean and variance of (2) are

$$\mathbf{E}Z(t) = \mu\alpha(t), \qquad \mathbf{D}Z(t) = \sigma^2\alpha(t) + \mu^2\beta^2(t). \tag{4}$$

Now we wish to find the mean and variance of the r.v. (3). At first, assume that $a(t)$, $t \geqslant 0$, is a step function $a(t) = a_l$, $t \in \Delta_l$, for sum intervals $\Delta_l = [s, t]$, $t \geqslant s$. In our case, $a(t) = e^{-\delta t}$, $\delta > 0$. We denote the integral sum for the integral $Z = \int_{t=0}^\infty a(t)\, dZ(t)$ by

$$Z^* = \sum_l a_l Z(\Delta_l), \tag{5}$$

where $Z(\Delta_l) = Z(t) - Z(s)$. By (4), and since $Z(t)$ increments are independent, we have

$$\mathbf{E}Z^* = \sum_l a_l \mathbf{E}Z(\Delta_l) = \mu \sum_l a_l \alpha(\Delta_l), \tag{6}$$

$$\mathbf{D}Z^* = \sum_l a_l^2 \mathbf{D}Z(\Delta_l) = \sigma^2 \sum_l a_l^2 \alpha(\Delta_l) + \mu^2 \sum_l a_l^2 \beta^2(\Delta_l), \tag{7}$$

where $\alpha(\Delta_l) = \alpha(t) - \alpha(s)$, $\beta^2(\Delta_l) = \beta^2(t) - \beta^2(s) > 0$. Let us denote,

$$N_{1,\delta} = \int_0^\infty e^{-\delta t}\, dN(t), \qquad N_{2,\delta} = \int_0^\infty e^{-2\delta t}\, dN(t). \tag{8}$$

The use of the integral sums $\sum_l a_l N(\Delta_l)$, $\sum_l a_l^2 N(\Delta_l)$ for the integrals (8), and (6) and (7), gives

$$\mathbf{E}Z_\delta = \mu\alpha_{1,*}, \qquad \mathbf{D}Z_\delta = \mathbf{E}Z_\delta^2 - (\mathbf{E}Z_\delta)^2 = \sigma^2\alpha_{2,*} + \mu^2\beta_{1,*}^2, \tag{9}$$

where $\alpha_{1,*}$, $\alpha_{2,*}$, and $\beta_{1,*}^2$ are means and the variance of r.v.s. (8):

$$\alpha_{1,*} = \mathbf{E}N_{1,\delta}, \qquad \alpha_{2,*} = \mathbf{E}N_{2,\delta}, \qquad \beta_{1,*}^2 = \mathbf{E}N_{1,\delta}^2 - (\mathbf{E}N_{1,\delta})^2. \tag{10}$$

Let

$$\bar{Z}_\delta = \frac{Z_\delta - \mathbf{E}Z_\delta}{\sqrt{2\mathbf{D}Z_\delta}}, \quad \mathbf{D}Z_\delta > 0. \tag{11}$$

be the normalized r.v. of the r.v. (3). In this paper, we are interested in the normal approximation for the distribution of (11) that takes into consideration large deviations both in the Cramér and the power Linnik zones in the case where cumulant method (see [9]) developed by Rudzkis, Saulis, Statulevičius (1978) (for the reference see in [9]) is used. In addition, we obtained nonuniform estimate in the Berry–Essen inequality. Observe, that in the paper we consider only the case where $\mu \neq 0$.

It should be noted that method of cumulants provided a way to obtain large deviation theorems for sums of independent and dependent r.v.s., polynomials forms, multiple stochastic integrals of random processes, and polynomial statistics in both the Cramér and the power Linnik zones. The monograph [9] addresses these issues.

Large deviation theorems in both the Cramér and the power Linnik zones, exponential inequalities, nonuniform estimates of the Berry–Essen inequality for a normalized compound Poisson process (the normalized sum (11), in case $N(t)$ is a homogeneous Poisson process) have been proved in [8]. Discounted version of the Berry–Essen theorem for the case of i.i.d. r.v.s. have been proved in [3]. Recently, Miao et al. [5] extended results of the paper from the i.i.d. r.v.s. case to the autoregressive process.

Since in this paper we are interested not only in the convergence to the normal distribution but also in a more accurate asymptotic analysis of the distribution function $F_{\tilde{Z}_N}(x)$, we must first find the suitable bound for the kth-order cumulants of (11). In order to obtain upper bounds for $\Gamma_k(\bar{Z}_\delta)$, we impose condition $(\bar{B}_\gamma)$ for the kth-order moments of X. Consequently, we say that the r.v. X with $0 < \sigma^2 < \infty$ satisfies condition $(\bar{B}_\gamma)$ if there exist constants $\gamma \geqslant 0$ and $K > 0$ such that

$$\left|\mathbf{E}(X-\mu)^k\right| \leqslant (k!)^{1+\gamma} K^{k-2} \sigma^2, \quad k = 3, 4, \dots . \tag{$\bar{B}_\gamma$}$$

Condition $(\bar{B}_\gamma)$ is a generalization of Bernstein's familiar condition. Condition $(\bar{B}_\gamma)$ ensures the existence of all order moments of the r.v. X. Using Lemma 3.1 in [9, p. 42], we take up the position that

Proposition 1. *If the r.v. X satisfies condition $(\bar{B}_\gamma)$, then*

$$\left|\Gamma_k(X)\right| \leqslant (k!)^{1+\gamma} M^{k-2} \sigma^2, \qquad M = 2\max\{\sigma, K\}, \quad k = 2, 3, \dots . \tag{12}$$

Along with the condition $(\bar{B}_\gamma)$, we impose condition for the kth-order cumulants of the process $N(t)$. Consequently, we assume that $N(t)$ satisfies condition (L): there exist constant $K_1 > 0$ such that

$$\left|\Gamma_k(N_t)\right| \leqslant (1/2) k! K_1^{k-2} \beta^2(t), \quad k = 2, 3, \dots . \tag{L}$$

Define the abbreviation $(a \vee b) = \max\{a, b\}$, $a, b \in \mathbb{R}$.

Lemma 1. *Suppose, the r.v. X with $\mu \neq 0$ and $0 < \sigma^2 < \infty$ fulfills condition $(\bar{B}_\gamma)$ and that $N(t)$ satisfies condition (L). Then, for $k = 3, 4, \dots,$*

$$\left|\Gamma_k(\bar{Z}_\delta)\right| \leqslant (k!)^{1+\gamma} / \left(3\Delta_*^{k-2}\right), \qquad \Delta_* = \sqrt{\mathbf{D}Z_\delta} / \Big(2\big(K_1|\mu| \vee \big(1 \vee \sigma/(2|\mu|)\big)M\big)\Big). \tag{13}$$

Since the accurate upper bounds (13) for the kth-order cumulants of the normalized sum $\tilde{Z}_N$ have been derived, to prove theorems of large deviations we have to use general lemma (see Lemma 2.3, Rudzkis, Saulis, Statulevičius, 1978 in [9, p. 18]), about large deviations for an arbitrary r.v. with zero mean and unit variance.

Let

$$\Delta_\gamma = c_\gamma {\Delta_*}^{1/(1+2\gamma)}, \qquad c_\gamma = (1/6)\left(\sqrt{2}/\left(6 \cdot 2^{1+\gamma}\right)^{1/(1+2\gamma)}\right), \quad \gamma \geqslant 0.$$

In addition, by $\Phi(\sqrt{2}x)$ we denote normal distribution function with zero mean and variance of $1/2$. Furthermore, we will use θ (with or without an index) to denote a value, not always the same, that does not exceed 1 in modulus.

Theorem 1. *Suppose the r.v. X with $0 < \sigma^2 < \infty$ fulfills condition $(\bar{B}_\gamma)$ and that $N(t)$ satisfies condition (L). Then in the interval $0 \leqslant x < \Delta_\gamma$, the ratios of large deviations*

$$\big(1 - F_{\bar{Z}_\delta}(x)\big)/\big(1 - \Phi(\sqrt{2}x)\big) = \exp\big\{L_\gamma(x)\big\}\big(1 + \theta_1 f(x)(x+1)/\Delta_\gamma\big),$$
$$F_{\bar{Z}_\delta}(-x)/\Phi(-\sqrt{2}x) = \exp\big\{L_\gamma(-x)\big\}\big(1 + \theta_2 f(x)(x+1)/\Delta_\gamma\big)$$

are valid, where

$$f(x) = \big(60\big(1 + 2\Delta_\gamma^2 \exp\big\{-(1 - x/\Delta_\gamma)\sqrt{\Delta_\gamma}\big\}\big)\big)/(1 - x/\Delta_\gamma),$$
$$L_\gamma(x) = \sum_{3\leqslant k<r} \tilde{\lambda}_k x^k + \theta_3\left(\frac{x}{\Delta_\gamma}\right)^3, \quad r = \begin{cases} 2 + \frac{1}{\gamma}, & \gamma > 0, \\ \infty, & \gamma = 0. \end{cases}$$

The coefficients $\tilde{\lambda}_k$ (expressed by cumulants of $\bar{Z}_\delta$ defined by (11)) coincide with the coefficients of the Cramér–Petrov series [7] given by the formula $\tilde{\lambda}_k = -b_{k-1}/k$, where the b_k are determined successively from the equations

$$\sum_{r=1}^{j} \frac{1}{r!}\Gamma_{r+1}(\bar{Z}_\delta) \sum_{j_1+\cdots+j_r=j,\, j_i\geqslant 1} \prod_{i=1}^{r} b_{j_i} = \begin{cases} 1, & j = 1, \\ 0, & j = 2, 3, \ldots. \end{cases}$$

In addition,

$$|\tilde{\lambda}_k| \leqslant \frac{4}{k}\left(\frac{32}{\Delta_*}\right)^{k-2}\big((k+1)!\big)^\gamma, \quad k = 2, 3, \ldots,$$
$$L_\gamma(x) \leqslant \frac{x^3}{x + 8\Delta_\gamma}, \qquad L_\gamma(-x) \geqslant -\frac{2x^3}{3\Delta_\gamma}.$$

Theorem 2. *Under the conditions of Theorem 1, the ratios*

$$\big(1 - F_{\bar{Z}_\delta}(x)\big)/\big(1 - \Phi(\sqrt{2}x)\big) \to 1, \qquad F_{\bar{Z}_\delta}(-x)/\Phi(-\sqrt{2}x) \to 1$$

hold for $x \geqslant 0$, $x = o(\Delta_*^{\nu(\gamma)})$, when $\Delta_* \to \infty$. Here $\nu(\gamma) = (1 + 2(1 \vee \gamma))^{-1}$.

Now let us assume, that where exist s moments of X and $N(t)$.

Theorem 3. *Let X with $0 < \sigma^2 < \infty$, satisfies condition $(\bar{B}_\gamma)$, for $k = 3, 4, \ldots, s$, and $N(t)$ satisfies condition (L), for $k = 2, 3, \ldots, s$. Then for all $x \geqslant 0$ and $m \leqslant s$, $m \geqslant 1$,*

$$\big|F_{\bar{Z}_\delta}(x) - \Phi(\sqrt{2}x)\big| \leqslant \frac{c(m,\gamma)\ln^{m/2}(\Delta_*/\sqrt{2})}{(1 + |\sqrt{2}x|^m)\Delta_*^{1/(1+2\gamma)}},$$

where $c(m,\gamma) = 2^{1/(2(1+2\gamma))}(1+2\gamma)^{-m/2}\big(252(6/\sqrt{2})^{1/(1+2\gamma)}c(m) \vee 12\cdot 4^\gamma(\sqrt{2}/6)^{2\gamma/(1+2\gamma)}\cdot m!\big)$, $c(m) = 2^{m/2}\big(5 + 2^{m/2}\sqrt{e/\pi}\,\Gamma((m+1)/2)\big)$.

1 Proofs of Lemma 1 and Theorems 1, 2, 3

Proof of Lemma 1. Since i.i.d. r.v.s. X_j, $j \geqslant 1$, and $N(t)$ are independent, the use of (11) in [4, p.258], gives the characteristic function of (2),

$$f_{Z(t)}(u) = \mathbf{E}e^{iuZ(t)} = f_{N(t)}\bigl(\ln f_X(u)/i\bigr), \quad u \in \mathbb{R}.$$

Thus, due to Lemma 5.6 in [7, p. 170], the k-th order cumulants

$$\begin{aligned} \mathit{\Gamma}_k(Z^*) &= \sum_l a_l^k \mathit{\Gamma}_k\bigl(Z(\mathit{\Delta}_l)\bigr) = \sum_l a_l^k \frac{d^k}{i^k du^k} \ln f_{N(\mathit{\Delta}_l)}\bigl(\ln f_X(u)/i\bigr)\big|_{u=0} \\ &= \sum_l a_l^k k! {\sum}_1^* \frac{\mathit{\Gamma}_m(N(\mathit{\Delta}_l))}{m_1! \cdot \dots \cdot m_k!} \prod_{j=1}^k \left(\frac{1}{j!}\mathit{\Gamma}_j(X)\right)^{m_j}, \quad k = 1, 2, \dots, \end{aligned}$$

of (5) exist if the kth-order cumulants of X and $N(t)$ exist. Here summation ${\sum}_1^*$ is taken over all non-negative integer solutions $(m_1, \dots, m_k)$ of the equation $m_1 + \dots + km_k = k$, $m_1 + \dots + m_k = m$, where $0 \leqslant m_1, \dots, m_k \leqslant k$, and $1 \leqslant m \leqslant k$.

Consequently, separating the summand of the sum ${\sum}_1^*$ in case where $m_1 = \dots = m_{k-1} = 0$, $m_k = 1$, and because of condition (L) together with the inequality (12), and inequalities (22)–(24) in [4, p. 261], we have that the inequality

$$\begin{aligned} &\bigl|\mathit{\Gamma}_k(Z^*)\bigr| \\ &\leqslant k! \sum_l a_l^k \left(\mathit{\Gamma}_1\bigl(Z(\mathit{\Delta}_l)\bigr)\bigl|\mathit{\Gamma}_k(X)\bigr|/k! + {\sum}_2^* \frac{\mathit{\Gamma}_{\tilde m}(N(\mathit{\Delta}_l))}{m_1! \cdot \dots \cdot m_{k-1}!} \prod_{j=1}^{k-1} \left(\frac{1}{j!}\bigl|\mathit{\Gamma}_j(X)\bigr|\right)^{m_j}\right) \\ &\leqslant (k!)^{1+\gamma} \left(2\left(K_1|\mu| \vee \left(1 \vee \frac{\sigma}{2|\mu|}\right)M\right)\right)^{k-2} \sum_l a_l^k \bigl(\alpha(\mathit{\Delta}_l)\sigma^2 + \beta^2(\mathit{\Delta}_l)\mu^2\bigr) \end{aligned}$$

is valid for $k = 3, 4, \dots$. Here ${\sum}_2^*$ is taken over all the non-negative integer solutions $(m_1, \dots, m_{k-1})$ of the equation $m_1 + \dots + (k-1)m_{k-1} = k$, $m_1 + \dots + m_{k-1} = \tilde m$, where $0 \leqslant m_1, \dots, m_{k-1} \leqslant k$, $2 \leqslant \tilde m \leqslant k$. Consequently,

$$\bigl|\mathit{\Gamma}_k(Z_\delta)\bigr| \leqslant (k!)^{1+\gamma} \mathbf{D}Z_\delta\bigl(2\bigl(K_1|\mu| \vee \bigl(1 \vee \sigma/(2|\mu|)\bigr)M\bigr)\bigr), \quad k = 3, 4, \dots. \tag{14}$$

To complete the proof of Lemma 1, it is sufficient to use (14), and then by noting that $\mathit{\Gamma}_k(\bar Z_\delta) = (2\mathbf{D}Z_\delta)^{-k/2}\mathit{\Gamma}_k(Z_\delta)$, $k = 2, 3, \dots$, we arrive at (13). □

Proof of Theorem 1. Clearly, $\bar Z_\delta$ satisfies Statulevičius' condition (see condition (S_γ), e.g. in [9, p. 16]) with the parameter, $\mathit{\Delta} := \mathit{\Delta}_*$. Accordingly, Lemma 2.3 in [9, p. 18] yields the assertion of Theorem 1. □

Proof of Theorem 2. Theorem 2 follows directly from Corrolary 3.1 in [9, p. 44]. □

Proof of Theorem 3. The proof of Theorem 3 is obtained by applying Achmedov's [1] lemma to the r.v. $\tilde Z_\delta = \sqrt{2}\bar Z_\delta$. □

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REZIUMĖ

Didžiųjų nuokrypių diskontavimo versija

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Tarkime $Z(t) = \sum_{j=1}^{N(t)} X_j$, $t \geqslant 0$, yra stochastinis procesas, čia X_j yra nepriklausomi, vienodai pasiskirstę atsitiktiniai dydžiai, o $N(t)$ yra sveikas, neneigiamas reikšmes įgyjantis, nepriklausomų pokyčių procesas. Laikoma, kad $N(t)$ ir X_j yra nepriklausomi. Šiame darbe yra nagrinėjama centruoto ir normuoto atsitiktinio dydžio $Z_\delta = \int_0^\infty e^{-\delta t} dZ(t)$, $\delta > 0$, pasiskirstymo funkcijos aproksimacija normaliuoju dėsniu, didžiųjų nuokrypių tiek Kramerio, tiek laipsninėse Liniko zonose. Taip pat, gautas netolygusis įvertis.

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